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Generalized SH-mode piezoelectric surface waves

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Generalized SH-mode piezoelectric surface waves

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I. INTRODUCTION

Acoustic wave propagation in solids first came to the attention of the electrical engineer in the late 1930's when a good delay line was needed for the development of the radar. In the early 1950's another engineering application became important, delay lines as storage devices for electronic computers. In 1961 Hutson et al. (17) developed a theory for acoustoelectric amplification and gave the engineer a new design component and made many new devices possible. Two more innovations since then have widened the engineering possibilities even more. These are piezoelectric surface waves (7,41), in which most of the acoustical energy is concentrated at the surface of a solid, and interdigital electrodes. In conjunction with one another these allow very efficient excitation and detection of acoustic waves and permit new filtering possibilities.

The entire November 1969 issue of IEEE Transactions on Microwave Theory and Techniques (20) was devoted to microwave acoustics and includes many good survey articles on possible device applications of piezoelectric surface waves. Because of the usefulness of surface waves, they have been the subject of considerable recent research and several devices have been marketed using surface waves. A bibliography on microwave ultrasonics has been compiled by Smith and Damon (37) which covers the research done on surface waves up to 1970.

In 1968 the existence of a new type of piezoelectric surface wave, the SH-mode piezoelectric surface wave, was demonstrated analytically by Bleustein (5) and Gulyaev (14). This surface wave has particle motion in the plane of the surface and perpendicular to the direction of propagation, and electric field components normal to the plane of the surface and parallel to the direction of propagation. There is no analogy to this wave in non-piezoelectric crystals and the advantages or disadvantages were not immediately obvious. Tseng (39), in an article which proved analytically the existence of the mode in a new crystal class, pointed out several possible advantages of the mode. These included: less sensitivity to surface defects, and less loss to harmonic generation at high powers. Since then a number of articles (1, 21, 29, 31, 43) have been written on theoretical and experimental amplification of the SH-mode piezoelectric surface wave and one author (29) reports an experimental gain much larger than is possible with Rayleigh waves. With these features and advantages the SH-mode promises to be very important in the future development of surface wave devices. Since the original theoretical work (5, 14), which was done for 6mm symmetry, other symmetries have been found to allow this mode. Tseng (39) reported the existence in cubic symmetry and later (40) in orthorhombic symmetry. Koerber and Vogel (24) expanded this list to include 4mm, 6, 4, $\bar{4}2m$, $\bar{6}m2$, $\bar{4}$; but the list is not complete and many good piezoelectric materials exist in other symmetry classes. Furthermore, in many of the above symmetry classes the solutions pertain to specific orientations. The problem of other orientations, for which the mode could potentially exist, is not solved by the treatments in the articles mentioned above.

To give the designer as much liberty as possible and to let him know what happens when the crystal is not cut precisely at some specified angle, a more general theory needs to be developed. This thesis will start with a treatment of the SH-mode in a very general setting. Then an exact solution will be derived for the crystal classes mentioned above, and finally, a variational treatment of the traveling wave problem will be used to obtain solutions for the heretofore unsolved problems.

II. THE SH-MODE

In isotropic elastic media a plane wave solution of the wave equation yields three solutions for the phase velocity (11). In seismological terminology these are the P (compressional), SV (vertically polarized shear), and SH (horizontally polarized shear) waves. In aeolotropic elastic media such simple descriptive names do not apply, but in general there are still three independent waves with mutually orthogonal displacement vectors, but these displacement vectors are not, in general, parallel or perpendicular to the direction of propagation (4, 22). However, certain alignments of propagation direction and crystalographic axes cause one of the displacement vectors to be perpendicular to the direction of propagation. This happens, for instance, when the propagation direction is parallel to a mirror plane of symmetry or perpendicular to a two-fold axis of symmetry. In these cases, one of the displacement vectors is normal to the plane of symmetry or parallel to the two-fold axis. The wave corresponding to this displacement vector will be referred to as an SH-mode, the same as in the isotropic case.

The boundary condition equations for a semi-infinite aeolotropic solid or, for an aeolotropic slab with air on both sides, or with air on one side and a solid on the other, do not couple the SH-mode to the other displacements if it has been decoupled by the above conditions; provided the mirror plane is perpendicular to the surfaces or the two-fold axis is parallel to the surfaces. The second case results in the transverse slab modes and the third results in Love waves (11). The first case is the surface wave problem and results in SH-mode surface waves. For an elastic half-space next to an air half-space these boundary conditions can't be satisfied (9). When the aeolotropic half-space is a piezoelectric material the problem is complicated by coupling to electric fields, but in certain cases the surface wave can exist and then is referred to as an SH-mode piezoelectric surface wave. This mode is the main topic of this thesis and will be treated in detail after a discussion of independent modes.

III. INDEPENDENT MODES

As mentioned in the last section the propagating waves decouple into indentifiable modes when the wave coordinates have certain relations to the crystalographic axes. It will now be shown formally when this decoupling occurs. The proof will include the effect of piezoelectricity. The equations of motion are

$$\rho \ddot{u}_i = T_{ij,j} \quad , \quad \vec{\nabla} \cdot \vec{D} = 0 \tag{1}$$

where ρ is mass density, u_i is particle displacement, T_{ij} is stress, a comma followed by an index denotes differentiation with respect to that space coordinate, a dot above a variable denotes differentiation with respect to time, and an arrow indicates vectors or vector operators. Index summation notation is used throughout. The equations of state (30, 33) for the solid are

$$\begin{aligned} T_{ij} &= C_{ijK\ell} S_{K\ell} - e_{Kij} E_K = C_{ijK\ell} u_{K,\ell} + e_{Kij} \phi_{,K} \\ D_i &= e_{ijK} S_{jK} + \epsilon_{ij} E_j = e_{ijK} u_{j,K} - \epsilon_{ij} \phi_{,j} \end{aligned} \tag{2}$$

Here $S_{K\ell}$ is strain, E_i is electric field intensity, D_i is electric flux density, ϕ is the quasi-electrostatic potential defined by $\vec{\nabla}\phi = -\vec{E}$, u_i is particle displacement and is related to S_{ij} by $S_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$, $C_{ijK\ell}$ is elastic stiffness, e_{Kij} is the piezoelectric tensor, and ϵ_{ij} is permittivity.

The general solutions of the motion equation are of the form

$$f = \exp\,(\Gamma_i x_i + j\omega t) \tag{3}$$

At this point the wave coordinate system must be specified. The direction of propagation will be the x_3 axis and in the case of a slab or a half-space the surfaces will be normal to the x_1 axis. The waves considered here will be uniform in the x_2 direction; that is $\Gamma_2 = 0$. This corresponds to bulk plane waves, guided waves in a slab, or straight crested surface waves in a half-space in the above coordinate system. Equations 1, 2, and 3 together with the definition of strain yields the following relation,

$$M_{ij}u_j = 0 \ , \quad i, j = 1 - 4 \tag{4}$$

where u_j, j = 1,2,3 is particle displacement, $u_4 = \phi$, and the elements of M_{ij} are

$$M_{11} = \omega^2\rho + \Gamma_1^{\,2}C_{11} + 2\Gamma_1\Gamma_3C_{15} + \Gamma_3^{\,2}C_{55}$$

$$M_{12} = \Gamma_1^{\,2}C_{16} + \Gamma_1\Gamma_3(C_{14} + C_{56}) + \Gamma_3^{\,2}C_{54}$$

$$M_{13} = \Gamma_1^{\,2}C_{15} + \Gamma_1\Gamma_3(C_{13} + C_{55}) + \Gamma_3^{\,2}C_{53}$$

$$M_{14} = \Gamma_1^{\,2}e_{11} + \Gamma_1\Gamma_3(e_{31} + e_{15}) + \Gamma_3^{\,2}e_{35}$$

$$M_{21} = M_{12}$$

$$M_{22} = \omega^2\rho + \Gamma_1^{\,2}C_{66} + 2\Gamma_1\Gamma_3C_{64} + \Gamma_3^{\,2}C_{44}$$

$$M_{23} = \Gamma_1^{\,2}C_{65} + \Gamma_1\Gamma_3(C_{63} + C_{45}) + \Gamma_3^{\,2}C_{43}$$

$$M_{24} = \Gamma_1^{\,2}e_{16} + \Gamma_1\Gamma_3(e_{36} + e_{14}) + \Gamma_3^{\,2}e_{34}$$

$$M_{31} = M_{13}$$

$$M_{32} = M_{23}$$

$$M_{33} = \omega^2\rho + \Gamma_1^{\,2}C_{55} + 2\Gamma_1\Gamma_3C_{53} + \Gamma_3^{\,2}C_{33}$$

$$M_{34} = \Gamma_1^{\;2} e_{15} + \Gamma_1\Gamma_3(e_{13} + e_{35}) + \Gamma_3^{\;2} e_{33}$$

$$M_{41} = M_{14}$$

$$M_{42} = M_{24}$$

$$M_{43} = M_{34}$$

$$M_{44} = -\Gamma_1^{\;2}\epsilon_{11} - 2\Gamma_1\Gamma_3\epsilon_{13} - \Gamma_3^{\;2}\epsilon_{33} \qquad (5)$$

The indices on the elastic constants and the last index on the piezoelectric constants are composites according to the scheme: $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $12 \rightarrow 6$, $13 \rightarrow 5$, $23 \rightarrow 4$. For the SH-mode (in this case u_2) to uncouple from the other displacements, the elements M_{12}, M_{21}, M_{23}, and M_{32} must be zero. In piezoelectric crystals it is also necessary for either (1) M_{24} and M_{42} or, (2) M_{14}, M_{34}, M_{41}, and M_{43} to be zero. These terms will be zero when the crystal is oriented so that its symmetry elements force the appropriate C's and e's to be zero. For instance, in an isotropic solid all the appropriate C's are zero for any orientation of the wave coordinates. In aeolotropic media the appropriate C's and e's will be zero whenever the direction of propagation is perpendicular to a two-fold axis of symmetry or parallel to a mirror plane of symmetry. To show this, consider the operator for a 180° rotation about the x_2 axis

$$O_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (6)$$

When this operator is used to transform the elastic constants from one coordinate system to a new coordinate system rotated 180°, the new elastic constants must be equal to the original ones in a crystal with a two-fold axis of symmetry parallel to the x_2 axis since the crystal is indistinguishable in the two coordinate systems. Primes are used to denote constants in the new coordinate system.

$$C_{ijK\ell} = C'_{ijK\ell} = 0_{im}0_{jn}0_{Ko}0_{\ell p}C_{mnop}$$

Since the operator has only diagonal components

$$C_{ijK\ell} = C'_{ijK\ell} = 0_{i}0_{j}0_{K}0_{\ell}C_{ijK\ell}$$

where $0_1 = 0_{11}$ et cetera. Any elastic constant with an odd number of one's and three's among its indices will be equal to its own negative and therefore vanishes.

The elements of the matrix M which were required to be zero in case (2) above all have an odd number of one's and three's in their indices and; therefore, the symmetry operator 0_{ij} leads to the required results. The same thing can be done for the symmetry operator which corresponds to a mirror plane parallel to the direction of propagation and the M's of case (1) would then be zero. Koerber (23) used the above method to work out all the decoupling conditions and their respective mode sets for surface waves in piezoelectric media. His result is that the mechanical decoupling is the same as that mentioned above; and in addition, when the symmetry element is a two-fold axis, the displacement component parallel to it is coupled to the electric potential. When the symmetry element is a mirror plane the displacement components in that plane are coupled to the electric potential. The former mode grouping, mode (1), will henceforth be called the piezoelectric SH-mode.

IV. THE PIEZOELECTRIC SH-MODE SURFACE WAVE

A. General Approach to the Problem

As was pointed out in the previous section, the least restrictive condition of symmetry which will allow a piezoelectric SH-mode is a two-fold axis. It follows that all piezoelectric SH-mode wave propagation problems can be represented by waves propagating in a crystal with only a two-fold axis of symmetry. This artifice will be used in this section and the next to solve many piezoelectric SH-mode problems.

The coordinate system shown in Figure 1 will be used throughout. Using the motion equations, Equations 1, the equations of state, Equation 2, and the waveform of Equation 3; Equation 4 is obtained; but, since it is assumed that there is a two-fold axis of symmetry parallel to the x_2 axis, M_{12}, M_{14}, M_{21}, M_{23}, M_{32}, M_{34}, M_{41}, and M_{43} of Equations 5 will be zero. This corresponds to two different modal groupings. The one treated here is the piezoelectric SH-mode so the matrix for it will be written separately.

$$[S_{ij}] = \begin{bmatrix} [\omega^2\rho+\Gamma_1^{\,2}C_{66}+2\Gamma_1\Gamma_3C_{46}+\Gamma_3^{\,2}C_{44}] & [\Gamma_1^{\,2}e_{16}+\Gamma_1\Gamma_3(e_{14}+e_{36})+\Gamma_3^{\,2}e_{34}] \\ [\Gamma_1^{\,2}e_{16}+\Gamma_1\Gamma_3(e_{14}+e_{36})+\Gamma_3^{\,2}e_{34}] & -[\Gamma_1^{\,2}\epsilon_{11}+2\Gamma_1\Gamma_3\epsilon_{13}+\Gamma_3^{\,2}\epsilon_{33}] \end{bmatrix} \tag{7}$$

For non-trivial solutions the determinant of the matrix $[S_{ij}]$ must be zero. This leads to the following algebraic equation,

$$(\omega^2\rho + \Gamma_1^{\,2}C_{66} + 2\Gamma_1\Gamma_3C_{46} + \Gamma_3^{\,2}C_{44})(\Gamma_1^{\,2}\epsilon_{11} + 2\Gamma_1\Gamma_3\epsilon_{13} + \Gamma_3^{\,2}\epsilon_{33})$$

$$+ (\Gamma_1^{\,2}e_{16} + \Gamma_1\Gamma_3(e_{14} + e_{36}) + \Gamma_3^{\,2}e_{34})^2 = 0 \tag{8}$$

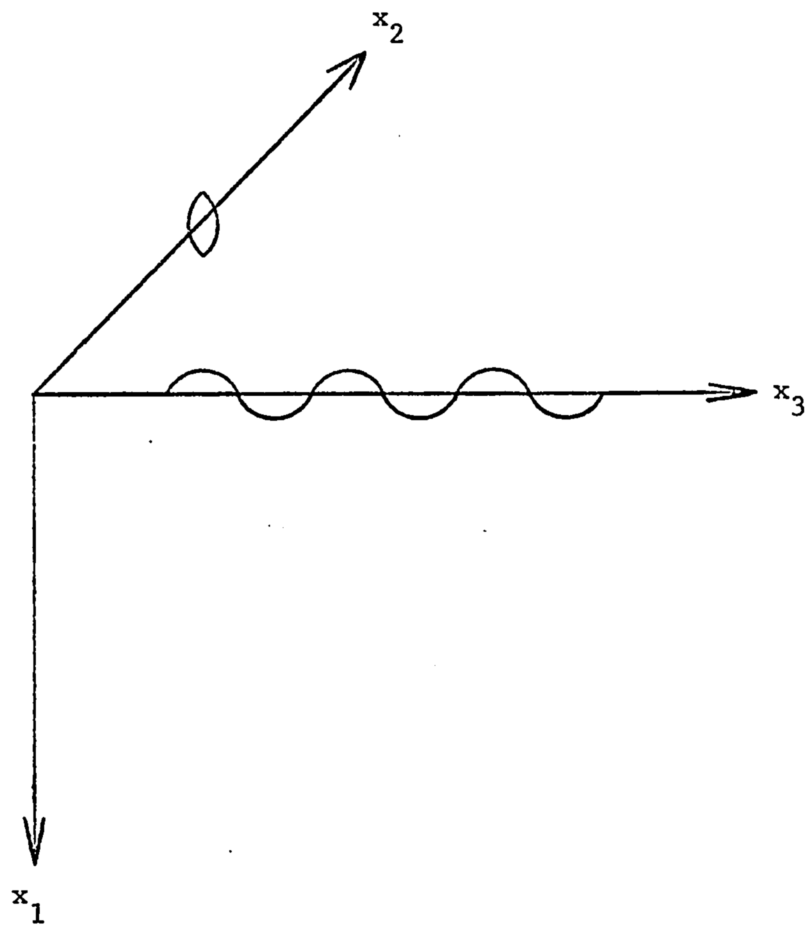

Figure 1. Coordinate system

At this point it will be assumed that wave is propagating in the x_3 direction, so $\Gamma_3 = -jK$ and $\Gamma_1 = j\,\Omega\,K$. The assumption for Γ_1 doesn't restrict the mathematical form of the wave in the x_1 direction but normalizes the dependence to K. Several ratios of material constants will be defined to simplify the equation.

$$\alpha_c = \frac{C_{46}}{C_{66}}, \quad \alpha_e = \frac{e_{14}+e_{36}}{e_{16}}, \quad \alpha_{e1} = \frac{e_{14}}{e_{16}}, \quad \alpha_{e2} = \frac{e_{36}}{e_{16}},$$

$$\delta = \frac{e_{34}}{e_{16}}, \quad \alpha_\epsilon = \frac{\epsilon_{13}}{\epsilon_{11}}, \quad \gamma = \frac{\epsilon_{33}}{\epsilon_{11}}, \quad \alpha = \frac{C_{44}}{C_{66}},$$

$$k^2 = \frac{e_{16}^2}{\epsilon_{11}C_{66}}, \quad V_S^2 = \frac{C_{66}}{\rho} \tag{9}$$

Substituting the wave's phase velocity parallel to the surface, V, for $\frac{\omega}{K}$, Equation 8 now becomes

$$\left(-\frac{V^2}{V_S^2} + \Omega^2 - 2\Omega\alpha_c + \alpha\right)\left(\Omega^2 - 2\Omega\alpha_\epsilon + \gamma\right) + k^2\left(\Omega^2 - \Omega\alpha_e + \delta\right)^2 = 0 \tag{10}$$

Letting $x = \frac{V^2}{V_S^2} - \alpha$ and expanding yields the polynomial form of the secular equation,

$$\Omega^4 - 2\,\frac{\alpha_c + \alpha_\epsilon + \alpha_e k^2}{1+k^2}\,\Omega^3 + \frac{\gamma - x + 4\alpha_c\alpha_\epsilon + 2\delta k^2 + \alpha_e^2 k^2}{1+k^2}\,\Omega^2 - 2\,\frac{\alpha_c\gamma - \alpha_\epsilon x + \delta\alpha_e k^2}{1+k^2}\,\Omega + \frac{\delta^2k^2 - \gamma x}{1+k^2} = 0 \tag{11}$$

By analogy coefficients a_0, a_1, a_2, and a_3 are defined,

$$\Omega^4 + a_3 \Omega^3 + a_2 \Omega^2 + a_1 \Omega + a_0 = 0 \tag{12}$$

B. Discussion of Roots

The roots of Equation 11 determine the character of the acoustic wave traveling in the x_3 direction. If it is a bulk wave then at least one solution of Ω will have to be zero. This is possible only when a_0 is zero.

$$a_0 = \frac{\delta^2 k^2 - \gamma x}{1 + k^2} = 0 \quad \text{or} \quad \gamma x = \delta^2 k^2 \quad \text{or} \quad \frac{V^2}{V_S^2} = \alpha + \frac{\delta^2 k^2}{\gamma} \tag{13}$$

This solution for the velocity is the piezoelectrically stiffened shear wave velocity. It can be rewritten as follows

$$V^2 = \frac{C_{44}}{\rho} + \frac{e_{34}^2}{\epsilon_{33}\rho} \tag{14}$$

A piezoelectrically stiffened elastic constant can be defined so that the equation will have the same form as a velocity equation in a non-piezoelectric solid.

$$\bar{C}_{44} = C_{44} + \frac{e_{34}^2}{\epsilon_{33}} \tag{15}$$

Then

$$V^2 = \frac{\bar{C}_{44}}{\rho} \tag{16}$$

This is the same $\overline{C}_{44}$ used by Bleustein (5) in his treatment of piezo-electric SH-mode surface waves in material of 6mm symmetry.

The other possible solutions include real r's, which correspond to propagation in the x_1 direction; and complex r's, which correspond to either surface waves or waves increasing in amplitude away from a surface depending on the sign of the imaginary part. In the case of real r's or complex r's with the proper sign on the imaginary part to give waves which increase in amplitude away from the surface, the solutions can correspond to guided waves in a slab.

C. Surface Waves

Since surface waves can only arise when the solutions of the secular equation give complex Ω's with non-zero imaginary parts, the left hand side of Equation 12 must not be zero for any real Ω. This leads to a condition on the coefficients of the equation as follows. The location of the minimum of the left hand side of Equation 12 as a function of real Ω's is a solution of

$$4\,\Omega^3 + 3\,a_3\,\Omega^2 + 2\,a_2\,\Omega + a_1 = 0 \qquad (17)$$

At this point an approximation is made. For actual materials a_1 and a_3 are small compared to unity and a_2 is of the order of unity, so a series solution can be found for Ω:

$$\Omega = -\frac{a_1}{2a_2} + \left(\frac{a_1}{2a_2}\right)^2 \left(\frac{a_1}{a_2^2} - \frac{3a_3}{2a_2}\right) + \begin{matrix}\text{term of}\\ \text{order}\end{matrix}\ \left(a_1^7 \text{ and } a_1^4 a_3^3\right) \qquad (18)$$

The first two terms of this series are an excellent approximation for actual material parameters. Using this solution in Equation 12, the height, h, of the minimum is found,

$$h = a_0 - \frac{a_1^2}{4a_2} + \text{terms of order } (a_1^4 \text{ and } a_1 a_3^3) \tag{19}$$

To insure no real solutions for Ω, h must always be greater than zero. So, to terms of order a_1^4 and $a_1^3 a_3$,

$$a_0 - \frac{a_1^2}{4a_2} > 0 \tag{20}$$

In many symmetries a_1 is zero and the above condition becomes exactly $a_0 > 0$, or

$$\frac{V^2}{V_S^2} < \alpha + \frac{\delta^2 k^2}{\gamma} \tag{21}$$

which is a statement that the surface wave velocity must be lower than the piezoelectrically stiffened bulk SH-mode wave velocity in the same direction. Tseng (39, 40) pointed this out in the cases he solved as did Bleustein (5). When the symmetry of the problem does not require a_1 to be zero, the surface wave velocity will be limited to an even lower velocity according to Equation 20.

If the above conditions are met surface waves can potentially exist and will be characterized by two of the four roots of Equation 12. Two roots of Equation 12 correspond to terms increasing in amplitude away from the surface; the other two correspond to terms decreasing in amplitude away from the surface. The latter will be retained for the surface wave characterization used to match the boundary conditions.

D. Boundary Conditions for Surface Waves

Figure 2 shows the surface of a piezoelectric material superimposed on the coordinate system of Figure 1. The propagation and two-fold axis directions are the x_3 axis and the x_2 axis respectively.

The conditions to be satisfied on the surface are; (1) the components of stress with a "1" among their indices must be zero, (2) the normal component of electric flux density must be continuous since there is no charge accumulation on the surface, and (3) the tangential components of electric field intensity must be continuous. The third condition is the same as requiring the quasi-electrostatic potential to be continuous because in the quasi-electrostatic approximation there is no x_2 component of electric field intensity and, since the disturbance is propagating in the x_3 direction, the spatial dependence of the fields in the x_3 direction will be the same above and below the surface. In other words $\phi^a = \phi^b$ is equivalent to $E_3^a = -\phi^a{}_{,3} = -\phi^b{}_{,3} = E_3^b$ since $\phi^a{}_{,3} = jK\phi^a = jK\phi^b = \phi^b{}_{,3}$. The condition to be satisfied at infinity is that all fields and displacements must go to zero. This condition is taken care of by the choice of the retained roots. Only roots which correspond to decreasing amplitude away from the surface are retained.

Symbolically, the boundary conditions are,

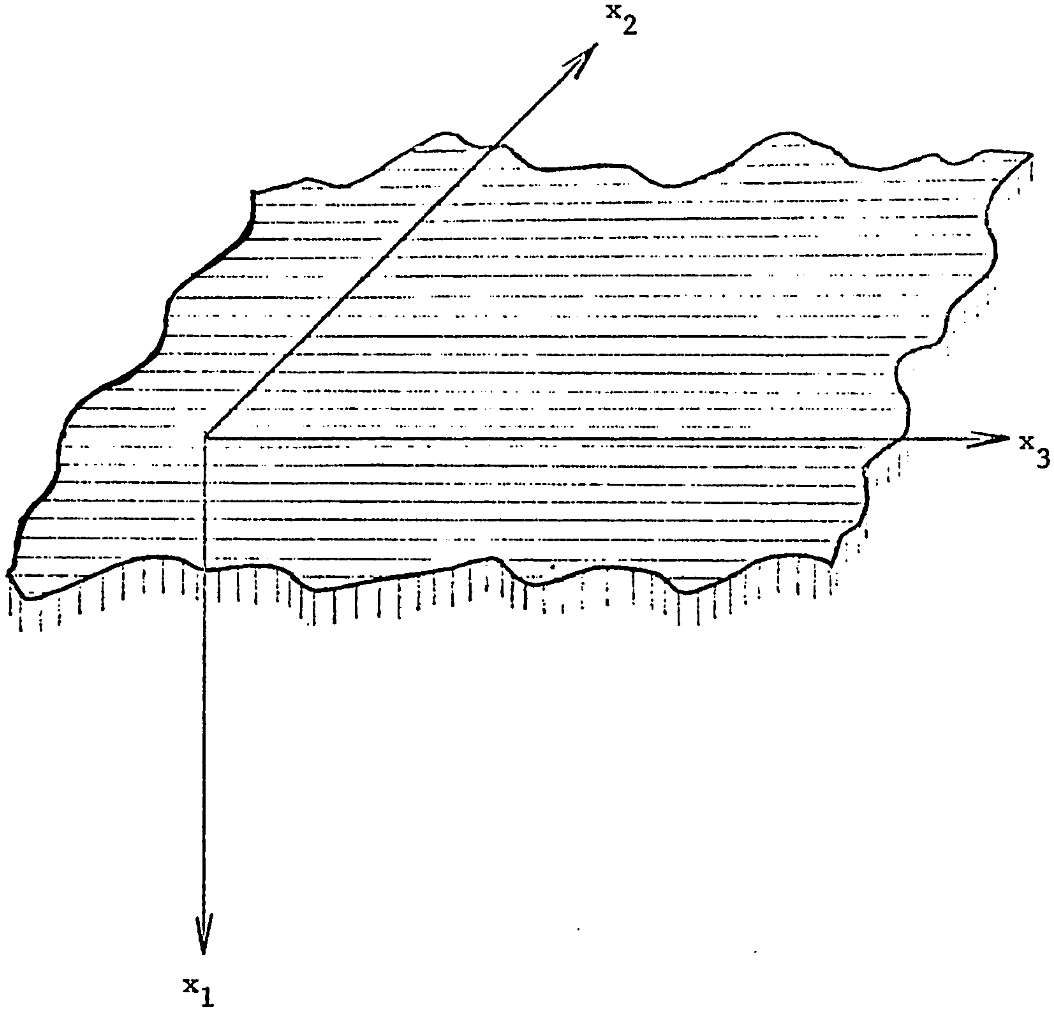

Figure 2. Relation of interface to coordinate system

$$T_{11} = T_{12} = T_{13} = 0$$
$$D_1^a = D_1^b \qquad (22)$$
$$\phi^a = \phi^b$$

The conditions $T_{11} = T_{13} = 0$ are identities in the case considered so the pertinent boundary conditions are

$$T_{12} = 0 \ , \quad D_1^a = D_1^b \ , \quad \phi^a = \phi^b \qquad (23)$$

These conditions, in terms of u_2 and ϕ, are

$$T_{12} = C_{46}u_{2,3} + C_{66}u_{2,1} + e_{16}\phi^a_{,1} + e_{36}\phi^a_{,3} = 0$$
$$D_1^a = e_{14}u_{2,3} + e_{16}u_{2,1} - \epsilon_{11}\phi^a_{,1} - \epsilon_{,3}\phi^a_{,3} = D_1^b = -\epsilon_o\phi^b_{,1} \qquad (24)$$
$$\phi^a = \phi^b$$

The form of u_2 and ϕ, as characterized by the two retained roots of Equation 12 in the form of Equation 3, will be

$$u_2 = u_1 e^{j(\Omega_1 Kx_1 - Kx_3 + \omega t)} + u_2 e^{j(\Omega_2 Kx_1 - Kx_3 + \omega t)}$$
$$\phi^a = \Phi_1 e^{j(\Omega_1 Kx_1 - Kx_3 + \omega t)} + \Phi_2 e^{j(\Omega_2 Kx_1 - Kx_3 + \omega t)} \qquad (25)$$
$$\phi^b = \bar{\Phi} e^{j(\beta Kx_1 - Kx_3 + \omega t)}$$

Either one of the equations represented by the matrix, 7, can be used to solve for the u coefficients in Equations 25 in terms of Φ. If the second is used, it results in

$$u_i = \frac{\epsilon_{11}\,\Omega^2_{(i)} - 2\epsilon_{13}\,\Omega_{(i)} + \epsilon_{33}}{e_{16}\,\Omega^2_{(i)} - (e_{14}+e_{36})\,\Omega_{(i)} + e_{34}}\,\Phi_{(i)} = R_i\,\Phi_{(i)} \tag{26}$$

The parentheses around the subscript means no sum is formed.

In the vacuum the only governing equation is

$$\vec{\nabla}\cdot\vec{D} = 0 \tag{27}$$

If the waveform $\phi^b = \bar{\Phi}\,e^{j(\beta K x_1 - K x_3 + \omega t)}$ is assumed Equation 27 leads to $\beta = -j$.

Equations 25 and 26 are now used in Equations 24 to obtain the following matrix form of the boundary conditions:

$$\begin{bmatrix} -C_{46}R_1+C_{66}\Omega_1R_1+e_{16}\Omega_1-e_{36} & -C_{46}R_2+C_{66}\Omega_2R_2+e_{16}\Omega_2-e_{36} & 0 \\ 1 & 1 & -1 \\ -e_{14}R_1+e_{16}\Omega_1R_1-\epsilon_{11}\Omega_1+\epsilon_{13} & -e_{14}R_2+e_{16}\Omega_2R_2-\epsilon_{11}\Omega_2+\epsilon_{13} & (-j\epsilon_0) \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \bar{\Phi} \end{bmatrix} = 0 \tag{28}$$

For non-trivial solutions to exist the determinant of this coefficient matrix must vanish. Using the definitions 10 and

$$r_i = \frac{\Omega^2_{(i)} - 2\alpha_\epsilon\Omega_{(i)} + \gamma}{\Omega^2_{(i)} - \alpha_e\Omega_{(i)} + \delta} = \frac{e_{16}}{\epsilon_{11}}R_i \tag{29}$$

this condition means that

$$\begin{vmatrix} -\alpha_c r_1+\Omega_1 r_1+k^2\Omega_1-\alpha_{e2}k^2 & -\alpha_c r_2+\Omega_2 r_2+k^2\Omega_2-\alpha_{e2}k^2 & 0 \\ 1 & 1 & -1 \\ -\alpha_{e1}r_1+\Omega_1 r_1-\Omega_1+\alpha_\epsilon & -\alpha_{e1}r_2+\Omega_2 r_2-\Omega_2+\alpha_\epsilon & -j\dfrac{1}{\epsilon_r} \end{vmatrix} = 0 \tag{30}$$

E. Exact Solutions

In this section the solution of Equations 11 and 30 will be given for crystals with high enough symmetry to allow an exact solution to be found. If Equation 11 reduces to a quadratic in r^2, the roots can be found analytically and, when substituted into the boundary condition of Equation 30, produce a reasonably simple algebraic equation for the surface wave velocity. The cases where this simplification occurs can be determined by an inspection of the charts of the allowable non-zero components of material properties (33). Table 1 shows some of the symmetries and directions of propagation which allow this simplification.

Equation 11 will be quadratic in r^2 if

$$\alpha_c = \alpha_\varepsilon = \alpha_e = 0 \tag{31}$$

It will then reduce to

$$\Omega^4 + \frac{\gamma - x + 2\delta k^2}{1 + k^2}\Omega^2 + \frac{\delta^2 k^2 - \gamma x}{1 + k^2} = 0 \tag{32}$$

and has solutions

$$\Omega = \pm\sqrt{\frac{-\gamma + x - 2\delta k^2 \pm \sqrt{(\gamma - x + 2\delta k^2)^2 - 4(1+k^2)(\delta^2 k^2 - \gamma x)}}{2(1+k^2)}} \tag{33}$$

The boundary condition equation, 30, will reduce to

$$\begin{vmatrix} \Omega_1 r_1 + k^2\Omega_1 - \alpha_{e2}k^2 & \Omega_2 r_2 + k^2\Omega_2 - \alpha_{e2}k^2 & 0 \\ 1 & 1 & -1 \\ \Omega_1 r_1 - \Omega_1 - \alpha_{e1} r_1 & \Omega_2 r_2 - \Omega_2 - \alpha_{e1} r_1 & -j\dfrac{1}{\varepsilon_r} \end{vmatrix} = 0 \tag{34}$$

Table 1. Cases which permit exact solutions

Symmetry Class	Direction of Propagation	Orientation of Sagittal Plane	References
6mm	1 to 6 fold axis	(0,0,1)	5, 14, 24
4mm	1 to 4 fold axis	(0,0,1)	24
$\bar{4}2m$	[1,1,0]	(0,0,1)	24
23, $\bar{4}3m$	[1,1,0]	(0,0,1)	24, 39, 40
23, $\bar{4}3m$	[1,0,1]	(0,1,0)	24, 39, 40
23, $\bar{4}3m$	[0,1,1]	(1,0,0)	24, 39, 40
$\bar{6}m2$	[1,0,0]	(0,1,0)	24
$\bar{6}m2$	[1,$\sqrt{3}$,0]	($\sqrt{3}$,-1,0)	24
$\bar{6}m2$	[1,-$\sqrt{3}$,0]	(-$\sqrt{3}$,-1,0)	24
$\bar{4}$	θ_c*	(0,0,1)	24

* θ_c must satisfy $\dfrac{e_{14}}{e_{15}} = \dfrac{2 \sin \theta_c \cos \theta_c}{\cos^2\theta_c - \sin^2\theta_c}$

Using Equation 29 and 33 in conjunction with Equation 34, the following expression for the surface wave velocity is obtained:

$$\frac{V^2}{V_S^2} = \alpha + \beta_1\beta_2 \left[(1+k^2)\epsilon_r(\beta_1+\beta_2)+1\right] - \epsilon_r\alpha_{e1}\alpha_{e2}k^2(\beta_1+\beta_2) \qquad (35)$$

where $\beta = -j\Omega$. If a surface wave solution is to exist, there must be a real solution for the velocity V which satisfies the condition of the inequality 21. The β's in Equation 35 are purely real and the assumption that $\alpha_e = \alpha_{e1} + \alpha_{e2} = 0$ means that the last term in Equation 37 is strictly negative. Using these facts and inequality 21 the condition,

$$-e_{14}e_{36} < \frac{e_{34}^2}{\epsilon_{r33}\sqrt{\gamma}} \qquad (36)$$

where $\epsilon_{r33} = \frac{\epsilon_{33}}{\epsilon_0}$, is demonstrated to be a condition for existence of the surface wave. Another limit on the surface wave velocity can be deduced from Equation 35. Since the right hand side of Equation 35 is the sum of three strictly positive terms, the velocity squared ratio can never be less than α. But, when $\frac{V^2}{V_S^2} = \alpha$, $V^2 = \frac{C_{44}}{\rho}$ and this is just the unstiffened bulk shear wave velocity in the same direction. So when the crystal symmetry and propagation direction simplify the equations according to Equations 31, the surface wave velocity is always between the piezoelectrically stiffened bulk shear wave velocity and the unstiffened bulk shear wave velocity in the same direction.

To show that a surface wave can exist, Equation 35 must be solved for the velocity in terms of material parameters only, and the β's must be real numbers satisfying $0 < \beta < \infty$. The solutions in this section will be those for an additional simplification of Equation 35 which can be solved exactly. Table 1 shows which symmetries and propagation directions produce this simplification. The simplification is

$$\delta = \gamma = \alpha = 1 \ , \quad \text{and} \quad \alpha_{e1} = \alpha_{e2} = 0 \tag{37}$$

This allows an explicit expression for the β's to be developed:

$$\beta_1 = 1, \quad \beta_2 = \sqrt{1 - \frac{V^2}{V_S^2\,(1 + k^2)}} \tag{38}$$

Equation 35 can now be solved explicitly for the velocity in terms of material parameters. The velocity squared ratio is

$$\frac{V^2}{V_S^2} = 1 + k^2 - \frac{k^4}{(1+\varepsilon_r)^2\,(1+k^2)} \tag{39}$$

which, in slightly different notation, is Bleustein's result. This result, however, is good for many more cases as outlined in Table 1. Most of these cases and a few others have been completely solved and categorized by Koerber and Vogel (24).

F. Approximate Solutions

When the symmetry of the material doesn't permit the simplifications shown in Table 1, Equations 11 and 30 are not algebraically manageable. To solve these more complicated problems a numerical technique can be used for specific orientations in specific materials, but this is not the best method from the design aspect. What is needed is a set of conditions which permit the existence of surface waves, a set of conditions which prohibit the existence of surface waves, and an approximation for the smallest decay constant, since it is an important parameter in determining the usefulness of the case considered.

To this end perturbational and variational techniques will be developed for surface waves. The perturbation technique (35), as developed by Slater, and presented for several electromagnetic problems by Harrington (15), and for elastic problems by Waldron (42), can be further developed to include the piezoelectric case as was done by Auld (3). A similar development for traveling waves will be presented here. The calculus of variation (6, 10, 12, 18, 34) can be used to derive stationary formulae for propagation constant or phase velocity in piezoelectric waveguides just as it is used in electromagnetic waveguides (15). The variational principle (18, 34) has been used extensively by Holland and EerNisse (16) and by Tiersten (38) to solve piezoelectric plate problems. A slightly different use will be made of the variational principle here to obtain a stationary formula for the phase velocity of a surface wave.

1. Perturbation

Using the notation and equations of Auld (3) for a solved problem, written with "0" subscripts, and an unsolved problem, written with no subscripts gives

$$-\nabla \times E_o = j\omega\mu H_o \qquad \nabla\cdot T_o = j\omega\rho V_o$$
$$\nabla \times H_o = j\omega D_o \qquad \nabla_s V_o = j\omega S_o \tag{40}$$

$$-\nabla \times E = j\omega\mu H \qquad \nabla\cdot T = j\omega\rho V$$
$$\nabla \times H = j\omega D \qquad \nabla_s V = j\omega S \tag{41}$$

Forming scalar products of the complex conjugates of the first through fourth of Equations 40, and the first through fourth of Equations 41 with H, E, V, T, H_o^*, E_o^*, and T_o^* respectively and regrouping the equations gives

$$-H\cdot\nabla \times E_o^* = -j\omega\mu H\cdot H_o^* \qquad V\cdot(\nabla\cdot T_o^*) = -j\omega\rho V\cdot V_o^*$$
$$E_o^*\cdot\nabla \times H = j\omega E_o^*\cdot D \qquad T_o^* : \nabla_s V = j\omega T_o^* : S \tag{42}$$

$$-H_o^*\cdot\nabla \times E = j\omega\mu H_o^*\cdot H \qquad V_o^*\cdot(\nabla\cdot T) = j\omega\rho V_o^*\cdot V$$
$$E\cdot\nabla \times H_o^* = -j\omega E\cdot D_o^* \qquad T : \nabla_s V_o^* = -j\omega T : S_o^* \tag{43}$$

All eight of these equations are added together and Green's Identities,

$$\nabla\cdot(A \times B) = B\cdot\nabla \times A - A\cdot\nabla \times B$$

and

$$\nabla\cdot(A\cdot C) = A\cdot(\nabla\cdot C) + C : \nabla_s A$$

where C is a rank two tensor and A and B are vectors, are applied to the resulting equation.

Integrating over the volume of a cross sectional slab of the surface wave device and using the divergence theorem on the left hand side, the following integral equation is obtained

$$\iint_S (H x E_o^* + H_o^* x E + V \cdot T_o^* + V_o^* \cdot T) \cdot da$$
$$= j\omega \iiint_V (E_o^* \cdot D - E \cdot D_o^* + T_o^* : S - T : S_o^*) d\tau \quad (44)$$

If it is assumed that a wave is propagating in the x_3 direction, the direction normal to the slab, the x_3 dependence of the dependent variables is e^{-jKx_3} and $e^{-jK_o x_3}$ and the x_3 dependence of the terms in the surface integral in Equation 44 is $e^{-j(K-K_o)x_3}$. If the slab is made infinitesimally thin Equation 46 can be rewritten

$$dx_3 \left\{ \frac{\partial}{\partial x_3} \iint_c (\hat{H} x \hat{E}_o^* + \hat{H}_o^* x \hat{E} + \hat{V} \cdot \hat{T}_o^* + \hat{V}_o^* \cdot \hat{T}) \cdot n_3 dS \right\}$$
$$= j\omega dx_3 \iint_c (\hat{E}_o^* \cdot \hat{D} - \hat{E} \cdot \hat{D}_o^* + \hat{T}_o^* : \hat{S} - \hat{T} : \hat{S}_o^*) dS \quad (45)$$

but, because of the x_3 dependence of the terms on the left,

$$-\frac{j(K-K_o)}{j\omega} \iint_c (\hat{H} x \hat{E}_o^* + \hat{H}_o^* x \hat{E} + \hat{V} \cdot \hat{T}_o^* + \hat{V}_o^* \cdot \hat{T}) \cdot n_3 dS$$
$$= -\left(\frac{1}{V_p} - \frac{1}{V_{po}}\right) \iint_c (\hat{H} x \hat{E}_o^* + \hat{H}_o^* x \hat{E} + \hat{V} \cdot \hat{T}_o^* + \hat{V}_o^* \cdot \hat{T}) \cdot n_3 dS$$
$$= \iint_c (\hat{E}_o^* \cdot \hat{D} - \hat{E} \cdot \hat{D}_o^* + \hat{T}_o^* : \hat{S} - \hat{T} : \hat{S}_o^*) dS \quad (46)$$

where C is an infinite plane normal to x_3. The caret means the x_3 dependence has been suppressed, and V_p and V_{po} are the unsolved and solved phase velocities respectively. Substituting from the equations of state, Equations 2, into the terms on the right on Equation 46 and assuming a material perturbation so that $C \rightarrow C + \Delta C$, $e \rightarrow e + \Delta e$, and $\epsilon \rightarrow \epsilon + \Delta\epsilon$.

$$-\left(\frac{1}{V_p} - \frac{1}{V_{po}}\right) \iint_c (\hat{H} \times \hat{E}_o^* + \hat{H}_o \times \hat{E} + \hat{V}\cdot\hat{T}_o^* + \hat{V}_o^*\cdot\hat{T})\cdot n_3 dS$$

$$= \iint_c \{\hat{E}_o^*\cdot[(e+\Delta e) : \hat{S} + (\epsilon+\Delta\epsilon)\cdot\hat{E}] - \hat{E}\cdot[e : \hat{S}_o^* + \epsilon\cdot\hat{E}_o^*]$$

$$+ \hat{S} : [C : \hat{S}_o^* - e\cdot\hat{E}_o^*] - \hat{S}_o^* : [(C+\Delta C) : \hat{S} - (e+\Delta e)\cdot\hat{E}]\} dS \tag{47}$$

Many terms on the right cancel, leaving

$$-\left(\frac{1}{V_p} - \frac{1}{V_{po}}\right) \iint_c (\hat{H} \times \hat{E}_o^* + \hat{H}_o^* \times \hat{E} + \hat{V}\cdot\hat{T}_o^* + \hat{V}_o^*\cdot\hat{T})\cdot n_3 dS$$

$$= \iint_c (\hat{E}_o^*\cdot\Delta e : \hat{S} + \hat{E}\cdot\Delta e : S_o^* + \hat{E}_o^*\cdot\Delta\epsilon\cdot\hat{E} - \hat{S}_o^* : \Delta C : \hat{S}) dS \tag{48}$$

Equation 48 is the general material perturbation formula for the phase velocity of the surface wave. To get the first order approximation from this formula, it is assumed that the unsolved fields are nearly equal to the solved fields and hence the solved fields can be used to substitute into both solved and unsolved fields in the formula. It then reduces to this form,

$$-\left(\frac{1}{V_p} - \frac{1}{V_{po}}\right) \iint_c [\mathrm{Re}\ (\hat{H}_o \times \hat{E}_o^*) + \mathrm{Re}\ (\hat{V}_o\cdot\hat{T}_o^*)]\cdot n_3 dS$$

$$= \iint_c [\mathrm{Re}\ (\hat{E}_o^*\cdot\Delta e : \hat{S}_o) + \hat{E}_o^*\cdot\Delta\epsilon\cdot\hat{E}_o - \hat{S}_o^* : \Delta C : \hat{S}_o] dS \tag{49}$$

This formula can be used to calculate changes in velocity due to changes in permittivity or elasticity; but, unfortunately, the only solved problem available yields zero for changes in the components of piezoelectricity e_{14} and e_{36} because the associated components of E and S are ninety degrees out of phase. These components of piezoelectricity are zero in the solved problem and hence no estimate of the change in velocity due to these components can be found. Since these components appear in most of the lower symmetry problems, a formula which includes them is desirable.

2. Stationary formulae through the calculus of variation

In text books which treat the calculus of variation, there is usually an illustration of the general principle of variation. It is shown how this principle leads to the equations of motion for a particular problem. This can be done in piezoelectric problems and the piezoelectric equations of motion can be derived. This is more than a curiosity since in many cases the integral form of the variational principle is also a stationary formula for some parameter of the system. Tiersten (38) and Holland and EerNisse (16) start with the variational principle to develop formulae for resonant frequencies of vibrating plates. A similar derivation will be used here to develop a stationary formula for propagation velocity.

a. General approach A variational treatment of the surface waveguide problem presents some problems which do not arise in the treatment of a vibrating finite slab. Unless sources are included the traveling wave will be infinite in extent in two dimensions. This means that volume integrals over the volume of interest will be unbounded. Furthermore, the operator for a traveling surface wave is not, in general, self-adjoint. These problems can be overcome with an artifice similar to that used by Morse and Feshbach (32) on a scattering problem and by Harrington (15) to get a variational solution for the inhomogeneously filled electromagnetic waveguide problem. The artifice works as follows. For every surface wave traveling in the positive x_3 direction there is a surface wave traveling in the negative x_3 direction which has a describing operator adjoint, with respect to the integrals involved, to that for the positive x_3 traveling wave. By using a positive and negative x_3 traveling wave in the integrals, the x_3 dependence can be eliminated. Since these waves are uniform in the x_2 direction, the integrals can now be written as bounded integrals and a variational principle can be derived.

The same physical arrangement and coordinate axes as in Section III will be used in this section. Let $f = \{u_i, \phi\}$ and $g = \{v_i, \psi\}$ be two independent field vectors and let $\mathcal{L}$ be the operator given by Equations 1 and 2. In other words

$$f = \{(C_{ijK\ell}u_{K,\ell j} + e_{\ell ij}\phi_{,\ell j}) \ , \ (e_{jK\ell}u_{K,\ell j} - \epsilon_{j\ell}\phi_{,\ell j})\}$$

$$= \{(T_{ij,j}) \ , \ (D_{j,j})\}$$

and

$$g = \{(C_{ijK\ell}V_{K,\ell j} + e_{\ell ij}\psi_{,\ell j}) \ , \ (e_{jK\ell}V_{K,\ell j} - \epsilon_{j\ell}\psi_{,\ell j})\}$$

$$= \{(T'_{ij,j}) \ , \ (D'_{j,j})\}$$

The form

$$< g, \mathcal{L} f > \; = \int_V g \mathcal{L} f dV \; , \tag{50}$$

where V is the volume enclosing the material, is chosen for the inner product and the conditions for the operator, $\mathcal{L}$, to be self-adjoint with respect to this inner product will be found.

$$\int_V g \mathcal{L} f dV = \int_V [V_i T_{ij,j} + \psi D_{j,j}] dV$$

$$= \int_V [V_i (C_{ijK\ell} u_{K,\ell j} + e_{\ell ij} \phi_{,\ell j}) + \psi (e_{jK\ell} u_{K,\ell j} - \epsilon_{j\ell} \phi_{,\ell j})] dV \tag{51}$$

Using the divergence theorem twice on the right hand integral of Equation 51, the following form is obtained:

$$\int_V g \mathcal{L} f dV = \int_V [u_K (C_{ijK\ell} V_{i,j\ell} + e_{jK\ell} \psi_{,j\ell}) + \phi (e_{\ell ij} V_{i,j\ell} - \epsilon_{j\ell} \psi_{,j\ell}] dV$$

$$+ \oint_S [V_i T_{ij} + \psi D_j] N_j dS - \oint_S [u_i T'_{ij} + \phi D'_j] N_j dS \tag{52}$$

where S is the surface enclosing V and N_j is the unit outward normal of S. The volume integral on the right in Equation 52 is equal to

$$\int_V f \mathcal{L} g dV$$

It can now be seen that the operator, $\mathcal{L}$, is self-adjoint if the difference of the two surface integrals in Equation 52 is zero.

The governing differential equation in the vacuum is $\vec{\nabla} \cdot \vec{D} = 0$. If the form of the electric field in the vacuum is $\exp(aKx_1 - jKx_3 + j\omega t)$, then the governing equation dictates that a = 1. Using this form of the electrical boundary conditions, the second and third of Equations 22, can be written

$$D_i N_i - \epsilon_o K\phi = 0 \qquad (53)$$

on the interface. The mechanical boundary condition is $T_{ij}N_j = 0$ on the interface. Rewriting the surface integrals of Equation 52 appropriately, the equation

$$\oint_s V_i T_{ij} N_j dS - \oint_s u_i T'_{ij} N_j dS + \oint_s \psi D_j N_j dS - \oint_s \phi D'_j N_j dS$$

$$- \oint_s \epsilon_o K\psi\phi dS + \oint_s \epsilon_o K\psi\phi dS = \oint_s V_i T_{ij} N_j dS - \oint_s u_i T'_{ij} N_j dS$$

$$+ \oint_s \psi(D_j N_j - \epsilon_o K\phi)dS - \oint \phi(d'_j N_j - \epsilon_o K\psi)dS \qquad (54)$$

is obtain.

Each one of the integrals on the right of Equation 54 is zero if the fields satisfy the boundary conditions at the interface and the contributions from sources at infinity can be neglected. This means that the operator, $\mathcal{L}$, is self-adjoint in the sense that for all functions f and g which satisfy the electrical and mechanical boundary conditions of Section III and which have no surface integral contributions due to sources at infinity (26),

$$< g, \mathcal{L}f > = < f, \mathcal{L}g >$$

This is too severe a restriction to put on the fields because if fields could be found to satisfy these restrictions the problem would be essentially solved. If the fields do not satisfy these restrictions the operator, $\mathscr{L}$, is not self-adjoint with respect to the choosen inner product. It is, as Gould (13) calls it, formally self-adjoint. Now instead of using a Lagrangian similar to that of Tiersten's (38) or Holland and EerNisse's (16), which require the operator to be self-adjoint in order for the first variation to be zero; a functional generated by use of the artifice mentioned above will be used.

b. <u>First acceptable Lagrangian</u> Two terms are formed, $(\rho\omega^2 u_i^+ u_i^- - u_{i,j}^+ T_{ij}^- - u_{i,j}^- T_{ij}^+ - \phi_{,j}^+ D_j^- - \phi_{,j}^- D_j^+)$ and $2\epsilon_o K\phi^+\phi^-$, where the superscript "+" indicates quantities for a wave traveling in the positive x_3 direction and "-" indicates quantities for the corresponding wave traveling in the negative x_3 direction. Each of the terms is a product of a positive x_3 going field and a corresponding negative x_3 going field. The x_3 dependence therefore cancels out. Also, since the surface waves are assumed to be uniform in the x_2 direction, there is no x_2 dependence.

Let F be a functional defined by

$$F = \int_o^\infty (\rho\omega^2 2u_i^+ u_i^- - u_{i,j}^+ T_{ij}^- - u_{i,j}^- T_{ij}^+ - \phi_{,j}^+ D_j^- - \phi_{,j}^- D_j^+) dx_i + 2\epsilon_o K\phi^+\phi^- \Big|_o^\infty \tag{55}$$

where x_1 is the direction normal to the interface. The first variation of F is

$$\delta F = \int_o^\infty (2\rho\omega^2 u_i^+ \, \delta u_i^- + 2\rho\omega^2 \, \delta u_i^+ \, u_i^- - \delta u_{i,j}^+ T_{ij}^- - u_{i,j}^+ \, \delta T_{ij}^-$$

$$- \delta\phi^+_{,j} D_j^- - \phi^+_{,j} \delta D_j^- - \delta u_{i,j}^- T_{ij}^+ - u_{i,j}^- \delta T_{ij}^+ - \delta\phi^-_{,j} D_j^+ - \phi^-_{,j} D_j^+) dx_1$$

$$+ (2\epsilon_o K \phi^+ \, \delta\phi^- + 2\epsilon_o K \phi^+ \phi^-) \Big|_o^\infty \qquad (56)$$

The fourth, sixth, eight, and tenth terms in the integral can be regrouped as in Equation 52:

$$u_{i,j}^+ \delta T_{ij}^- + \phi^+_{,j} \delta D_j^- + u_{i,j}^- \delta T_{ij}^+ + \phi^-_{,j} \delta D_j^+$$

$$= \delta u_{i,j}^- T_{ij}^+ + \delta\phi^-_{,j} D_j^+ + \delta u_{i,j}^+ T_{ij}^- + \delta\phi^+_{,j} D_j^-$$

All but the first two terms are integrated by parts and the result rearranged to form

$$\delta F = 2 \int_o^\infty [\delta u_i^- (\rho\omega^2 u_i^+ + T_{ij,j}^+) + \delta u_i^+ (\rho\omega^2 u_i^- + T_{ij,j}^-)$$

$$+ \delta\phi^- (D_{j,j}^+) + \delta\phi^+ (D_{j,j}^-)] dx_1 - 2[\delta u_i^- (T_{ij}^+ N_j)$$

$$+ \delta u_i^+ (T_{ij}^- N_j) + \delta\phi^- (D_j^+ N_j - \epsilon_o K \phi^+) + \delta\phi^+ (D_j^- N_j - \epsilon_o K \phi^-)] \Big|_o^\infty$$

(57)

For this to be zero it is necessary for each of the terms in parentheses to be zero, but that is just the statement of the original problem involving the equations of motion, Equations 1, and the boundary conditions, Equations 53. In other words the functional F is stationary (has a zero first variation) around those fields that solve the original boundary value problem.

The value of F when the first variation is zero can be found by following the steps from Equation 55 through 57 only without taking the first variation and the result, an equation similar to Equation 58, will show that this value is zero. Then by rearranging Equation 55 a formula for $\rho\omega^2$ is suggested.

$$\rho\omega^2 = R_{\text{stationary}},$$

where

$$R = \frac{\int_0^\infty (u^+_{i,j}T^-_{ij} + u^-_{i,j}T^+_{ij} + \phi^+_{,j}D^-_j + \phi^-_{,j}D^+_j)dx_1 - 2\epsilon_o K\phi^+\phi^- \Big|_o^\infty}{\int_0^\infty 2u^+_i u^-_i \, dx_1} \tag{58}$$

To prove that $\rho\omega^2 = R_{\text{stationary}}$ it is only necessary to note that

$$\delta R = \frac{D\delta N - N\delta D}{D^2} = \frac{D\delta N - (\rho\omega^2 D - F)\delta D}{D^2}$$

$$= \frac{D(\delta N - \rho\omega^2\delta D) + F\delta D}{D^2} = \frac{-D\delta F + F\delta D}{D^2} \tag{59}$$

where N and D are the numerator and denominator of R respectively. It was noted earlier that δF and F are both zero when the fields are described by the original boundary value problem. Therefore, a stationary formula for $\rho\omega^2$ has been found and, as will be shown later, a stationary formula for wave velocity can be found from this.

<u>c. Simplified Lagrangian</u> The stationary form for $\rho\omega^2$ presented in the last section is good for any surface wave problem where the space on one side of the interface is isotropic, homogeneous, and massless,

and the other side is an arbitrarily aeolotropic, piezoelectric, homogeneous medium. This work is directed only to the SH-mode and in this case some simplifications can be made in Equations 58. First, the nature of the secular equation, Equation 12, determines the form of the surface wave. For instance u_2^+ has the form

$$u_2^+ = (u_1^+ e^{-\beta_1 Kx_1} + u_2^+ e^{-\beta_2 Kx_1}) e^{-j(Kx_3 - \omega t)}$$

and ϕ^+ has the form (60)

$$\phi^+ = (\Phi_1^+ e^{-\beta_1 Kx_1} + \Phi_2^+ e^{-\beta_2 Kx_1}) e^{-j(Kx_3 - \omega t)}$$

To be exact, β_1 and β_2 would have to be the two solutions with positive real parts of the following equation, which is similar to Equation 11,

$$\beta^4 + 2j \frac{\alpha_c + \alpha_\epsilon + \alpha_e k^2}{1 + k^2} \beta^3 - \frac{\gamma - x + 4\alpha_c \alpha_\epsilon + 2\delta k^2 + \alpha_e^2 k^2}{1 + k^2} \beta^2$$

$$- 2j \frac{\alpha_c \gamma - \alpha_\epsilon x + \delta \alpha_e k^2}{1 + k^2} \beta + \frac{\delta^2 k^2 - \gamma x}{1 + k^2} = 0 \qquad (61)$$

The last two terms in the integral in the numerator of Equation 58 are integrated by parts and R becomes

$$R = \frac{\int_0^\infty u_{i,j}^+ T_{ij}^- + u_{i,j}^- T_{ij}^+ - \phi^+ [D_{i,j}^-] - \phi^- [D_{i,i}^+]) dx_1 - \left. - (\phi^+ [D_j^- N_j - \epsilon_o K \phi^-] + \phi^- [D_j^+ N_j - \epsilon_o K \phi^+]) \right|_0^\infty}{2 \int_0^\infty u_i^+ u_i^- \, dx_1}$$

From this form it is seen that R can be written as

$$R = \frac{\int_o^\infty (u^+_{i,j}T^-_{ij} + u^-_{i,j}T^+_{ij})dx_1}{2\int_o^\infty u^+_i u^-_i \, dx_1} \tag{62}$$

if the terms in brackets are zero; that is, if $\vec{\nabla}\cdot\vec{D} = 0$ and the electrical boundary conditions are satisfied. Since the form of the fields is known these conditions can be met by requiring that

$$\Phi^+_1 = \frac{e_{16}}{\epsilon_{11}} \ell^+_1 U^+_1 , \quad \Phi^+_2 = \frac{e_{16}}{\epsilon_{11}} \ell^+_2 U^+_2 , \quad \Phi^-_1 = \frac{e_{16}}{\epsilon_{11}} \ell^-_1 U^-_1 , \quad \Phi^-_2 = \frac{e_{16}}{\epsilon_{11}} \ell^-_2 U^-_2$$

where

$$\ell^+_1 = \frac{\beta_1^2 + j\alpha_e\beta_1 - \delta}{\beta_1^2 + 2j\alpha_\epsilon\beta_1 - \gamma} , \quad \ell^+_2 = \frac{\beta_2^2 + j\alpha_e\beta_2 - \delta}{\beta_2^2 + 2j\alpha_\epsilon\beta_2 - \gamma} \tag{63}$$

ℓ^-_1 = complex conjugate of ℓ^+_1, and ℓ^-_2 = complex conjugate of ℓ^-_2 ,

and

$$U^+_1 = A^+U^+_2 , \quad U^-_1 = A^-U^-_2$$

where

$$A^+ = \frac{je_{14} + e_{16}\beta_2 - (\epsilon_{11}\beta_2 + j\epsilon_{13} + \epsilon_o)\frac{e_{16}}{\epsilon_{11}}\ell^+_2}{-je_{14} - e_{16}\beta_1 + (\epsilon_{11}\beta_1 + j\epsilon_{13} + \epsilon_o)\frac{e_{16}}{\epsilon_{11}}\ell^+_1} \tag{64}$$

and A^- = complex conjugate of A^+ .

Equations 63 come from a modification of Equation 26 and insure that $\vec{\nabla}\cdot\vec{D} = 0$. Equations 63 come from a modification of Equation 28 and insure that the electrical boundary conditions are satisfied.

d. Comparison and demonstration that the simplified Lagrangian is a relative minimum It can be noted that Equation 58 corresponds to the variational problem of Bolza (6) with variable end points and no side conditions. Equation 62 with the electrical condition that $\vec{\nabla}\cdot\vec{D} = 0$ and that the electrical boundary conditions be satisfied, is the variational problem of Lagrange (6). The steps from Equation 58 to 62 just transform it from one type of problem to another (6) but Equation 61 is simpler to apply. Equation 58 has the disadvantage that its expression for R is neither a maximum nor a minimum where the first variation is zero. The value of R given in Equation 62 is a relative minimum where the first variation is zero because of the way the field of admissible functions is restricted by the side condition, $\vec{\nabla}\cdot\vec{D} = 0$. This can be shown as follows. First, the formula for R, Equation 62, can be rewritten to include a quantity which is zero by the side condition, $\vec{\nabla}\cdot\vec{D} = 0$. This resembles the procedure for using the Lagrange multiplier rule (34) but here the multiplier will be ϕ instead of arbitrary as usual.

$$R = \frac{\int_o^\infty (u^+_{i,j}T^-_{ij} + u^-_{i,j}T^+_{ij} + \phi^+ D^-_{K,K} + \phi^- D^+_{K,K})dx_1}{2\int_o^\infty u^+_i u^-_i \, dx_1} \tag{65}$$

Integrating the last two terms in the numerator by parts and applying the electrical boundary condition gives

$$R = \frac{\left. \int_o^\infty (u^+_{i,j}T^-_{ij} + u^-_{i,j}T^-_{ij} - \phi^+_{,K}D^-_K - \phi^-_{,K}D^+_K)dx_1 + 2\phi^+\varepsilon_o K\phi^- \right|_o^\infty}{2\int_o^\infty u^+_i u^-_i \, dx_1} \tag{66}$$

The second variation of this can be found by a process similar to that used to give the first variation, and by repeatedly using the electrical side and boundary conditions the following form is obtained

$$\delta^2 R = \frac{\int_o^\infty (\delta u^+_{i,j}\delta T^-_{ij}+\delta u^-_{i,j}\delta T^+_{ij}-\delta\phi^+_{,K}\delta D^-_K-\delta\phi^-_{,K}\delta D^+_K)dx_1+\epsilon_o K\delta\phi^+\delta\phi^- 2\Big|_o^\infty}{2\int_o^\infty \delta u^+_i\, \delta u^-_i\, dx_1} \tag{67}$$

or

$$\delta^2 R = \frac{\int_o^\infty (\delta u^+_{i,j} C_{ijK\ell}\delta u^-_{K,\ell}+\delta\phi^+_{,K}\epsilon_{K\ell}\delta\phi^-)dx_1+\epsilon_o K\delta\phi^+\delta\phi^- \Big|_o^\infty}{\int_o^\infty \delta u^+_i\, \delta u^-_i\, dx_1} \tag{68}$$

To show that this is strongly positive, a process similar to Legendre's transformation (34) will be used. First, Equation 68 will be expanded, realizing that for the SH-mode only one component of u_i is present and due to earlier arguments only "2" symmetry need be considered.

$$\delta^2 R = \frac{N_2}{D_2} \tag{69}$$

where

$$N_2 = \int_o^\infty (C_{66}\delta u^+_{2,1}\delta u^-_{2,1}-jKC_{46}\delta u^-_{2,1}\delta u^+_2+jKC_{46}\delta u^-_2\delta u^+_{2,1}$$

$$+ K^2C_{44}\delta u^+_2\delta u^-_2+\delta\phi^+_{,1}\epsilon_{11}\delta\phi^-_{,1}-jK\epsilon_{13}\delta\phi^+_{,1}\delta\phi^-$$

$$+ jK\epsilon_{13}\delta\phi^+\delta\phi^-_{,1}+\epsilon_{33}\delta\phi^+\delta\phi^-)dx_1+\epsilon_o K\delta\phi^+\delta\phi^- \Big|_o^\infty$$

and

$$D_2 = \int_o^\infty \delta u_2^+ \, \delta u_2^- \, dx_1$$

To N_2 of Equation 69 will be added the following term

$$\int_o^\infty [C_{66} \frac{d}{dx_1} (f\delta u_2^+ \, \delta u_2^-) + \epsilon_{11} \frac{d}{dx_1} (g\delta\phi^+ \, \delta\phi^-)] dx_1$$

This term will be zero provided f and g are chosen so that they are zero at $x_1 = 0$ and grow slower than exponentially as x_1 goes to infinity. With this addition Equation 69 can be rearranged and written as follows,

$$\frac{\delta^2 R}{C_{66} K^2} = \frac{N_3}{D_3} \tag{70}$$

where

$$N_3 = \int_o^\infty \{[\delta u^+_{2,1} + (f - j\alpha_c)\delta u_2^+][\delta u^-_{2,1} + (f + j\alpha_c)\delta u_2^-]$$

$$+ (f_{11} + \alpha - f^2 - \alpha_c^{\,2}) \, \delta u_2^+ \, \delta u_2^-$$

$$+ \frac{\epsilon_{11}}{C_{66}} [\delta\phi^+_{,1} + (g - j\alpha_\epsilon) \, \delta\phi^+][\delta\phi^-_{,1} + (g + j\alpha_\epsilon) \, \delta\phi^-]$$

$$+ \frac{\epsilon_{11}}{C_{66}} (g_{,1} + \gamma - g^2 - \alpha_\epsilon^2) \, \delta\phi^+ \, \delta\phi^- \} \, d(Kx_1)$$

$$+ \epsilon_o \delta\phi^+ \, \delta\phi^- \Big|_o^\infty$$

and

$$D_3 = \int_o^\infty \delta u_i^+ \, \delta u_i^- \, d(Kx_1)$$

The derivatives and integrals in Equation 70 are with respect to Kx_1. The two Riccati equations

$$f_{,1} + \alpha - f^2 - \alpha_c^{\ 2} = 0 \quad \text{and} \quad g_{,1} + \gamma - g^2 - \alpha_\epsilon^{\ 2} = 0$$

have solutions

$$f = -\sqrt{\alpha - \alpha_c^{\ 2}}\ \tanh \sqrt{\alpha - \alpha_c^{\ 2}}\ x_1$$

and

$$g = -\sqrt{\gamma - \alpha_\epsilon^{\ 2}}\ \tanh \sqrt{\gamma - \alpha_\epsilon^{\ 2}}\ x_1$$

provided $\gamma > \alpha_\epsilon^{\ 2}$ and $\alpha > \alpha_c^{\ 2}$.

These functions for f and g satisfy the conditions placed on them earlier and the inequalities above are satisfied for every known material. By inspection of the motion equation it can be shown that for every u^+ there is a u^- such that $u^- = (u^+)^*$ and similarly for ϕ. This means that the products with brackets in them and the last term in N_3 and the denominator of Equation 70 are magnitudes of complex numbers and therefore non-negative. Now, all the criteria have been satisfied for a sufficiency theorem (29) to apply and the existence of a relative minimum is assured.

e. Expansion of the simplified Lagrangian into a useable form

Equation 62 will now be expanded and divided by C_{66} to obtain

$$\frac{R}{C_{66}} = \frac{N_4}{D_4}$$

where

$$N_4 = \int_o^\infty [u^+_{,1}u^-_{,1} - jKu^+ u^-_{,1}\alpha_c + jKu^+_{,1}u^- \alpha_c + K^2u^+ u^- \alpha$$

$$+ \frac{1}{2} (u^+_{,1}\phi^-_{,1} \frac{e_{16}}{C_{66}} - jKu^+\phi^-_{,1} \frac{e_{14}}{C_{66}} + jKu^+_{,1}\phi^- \frac{e_{36}}{C_{66}} + K^2u^+\phi^- \frac{e_{34}}{C_{66}}$$

$$+ u^-_{,1}\phi^+_{,1} \frac{e_{16}}{C_{66}} - jKu^-_{,1}\phi^+ \frac{e_{36}}{C_{66}} + jKu^-\phi^+_{,1} \frac{e_{14}}{C_{66}} + K^2u^-\phi^+ \frac{e_{34}}{C_{66}})]dx_1$$

and

$$D_4 = \int_o^\infty u^+ u^- dx_1$$

Here, u is u_2 with the x_3 dependence removed and ϕ has no x_3 dependence. Now, substituting from Equations 63 and 64 and dividing by K^2 gives

$$\frac{R}{K^2C_{66}} = \frac{\int_o^\infty [\alpha Q_1 + Q_2 + j\alpha_c Q_3 + \frac{1}{2} k^2 (Q_4 + j\alpha_{e1}Q_5 + j\alpha_{e2}Q_6 + \delta Q_7)] dx_1}{\int_o^\infty Q_1 \, dx_1} \tag{71}$$

where

$$Q_1 = (A^+ e^{-\beta_1 Kx_1} + e^{-\beta_2 Kx_1})(A^- e^{-\beta_1^* Kx_1} + e^{-\beta_2^* Kx_1})$$

$$Q_2 = (A^+\beta_1 e^{-\beta_1 Kx_1} + \beta_2 e^{-\beta_2 Kx_1})(A^-\beta_1^* e^{-\beta_1^* Kx_1} + \beta_2^* e^{-\beta_2^* Kx_1})$$

$$Q_3 = (A^+ e^{-\beta_1 Kx_1} + e^{-\beta_2 Kx_1})(A^-\beta_1^* e^{-\beta_1^* Kx_1} + \beta_2^* e^{-\beta_2^* Kx_1})$$
$$- (A^+\beta_1 e^{-\beta_1 Kx_1} + \beta_2 e^{-\beta_2 Kx_1})(A^- e^{-\beta_1^* Kx_1} + e^{-\beta_2^* Kx_1})$$

$$Q_4 = (A^+\beta_1 e^{-\beta_1 Kx_1} + \beta_2 e^{-\beta_2 Kx_1})(A^-\ell_1^-\beta_1^* e^{-\beta_1^* Kx_1} + \ell_2^-\beta_2^* e^{-\beta_2^* Kx_1})$$
$$(A^+\ell_1^+\beta_1 e^{-\beta_1 Kx_1} + \ell_2^+\beta_2 e^{-\beta_2 Kx_1})(A^-\beta_1^* e^{-\beta_1^* Kx_1} + \beta_2^* e^{-\beta_2^* Kx_1}$$

$$Q_5 = (A^+ e^{-\beta_1 Kx_1} + e^{-\beta_2 Kx_1})(A^- \ell_1^- \beta_1^* e^{-\beta_1^* Kx_1} + \ell_2^- \beta_2^* e^{-\beta_2^* Kx_1})$$
$$- (A^+ \ell_1^+ \beta_1 e^{-\beta_1 Kx_1} + \ell_2^+ \beta_2 e^{-\beta_2 Kx_1})(A^- e^{-\beta_1^* Kx_1} + e^{-\beta_2^* Kx_1})$$

$$Q_6 = (A^+ \ell_1^+ e^{-\beta_1 Kx_1} + \ell_2^+ e^{-\beta_2 Kx_1})(A^- \beta_1^* e^{-\beta_1^* Kx_1} + \beta_2^* e^{-\beta_2^* Kx_1})$$
$$- (A^+ \beta_1 e^{-\beta_1 Kx_1} + \beta_2 e^{-\beta_2 Kx_1})(A^- \ell_1^- e^{-\beta_1^* Kx_1} + \ell_2^- e^{-\beta_2^* Kx_1})$$

$$Q_7 = (A^+ e^{-\beta_1 Kx_1} + e^{-\beta_2 Kx_1})(A^- \ell_1^- e^{-\beta_1^* Kx_1} + \ell_2^- e^{-\beta_2^* Kx_1})$$
$$+ (A^+ \ell_1^+ e^{-\beta_1 Kx_1} + \ell_2 e^{-\beta_2 Kx_1})(A^- e^{-\beta_1^* Kx_1} + e^{-\beta_2^* Kx_1})$$

Equation 71 can be easily integrated. The result is

$$\frac{R}{K^2 C_{66}} - \alpha = \frac{N_5}{D_5} \tag{72}$$

where

$$N_5 = \{|A^+|^2|\beta_1|^2 \frac{1}{2 \text{ Re } \beta_1} + 2 \text{ Re } [A^+ \frac{\beta_1\beta_2^*}{\beta_1+\beta_2^*}] + |\beta_2|^2 \frac{1}{2 \text{ Re } \beta_2}\}$$
$$+ \alpha_c\{|A^+|^2 \frac{\text{Im } \beta_1}{\text{Re } \beta_1} + 2 \text{ Im } [A^+ \frac{\beta_1-\beta_2^*}{\beta_1+\beta_2^*}] + \frac{\text{Im } \beta_2}{\text{Re } \beta_2}\}$$
$$+ \frac{1}{2} k^2 \{|A^+|^2|\beta_1|^2 \frac{\text{Re } \ell_1^+}{\text{Re } \beta_1} + 2 \text{ Re } [A^+\beta_1\beta_2^* \frac{\ell_1^+ + \ell_2^-}{\beta_1+\beta_2^*}] + |\beta_2|^2 \frac{\text{Re } \ell_2^+}{\text{Re } \beta_2}$$

$$+ \frac{1}{2} k^2\alpha_{e1}\{|A^+|^2 \frac{\text{Im } \ell_1^+\beta_1}{\text{Re } \beta_1} + 2 \text{ Im } [A^+ \frac{\ell_1^+\beta_1 - \ell_2^-\beta_2^*}{\beta_1+\beta_2^*}] + \frac{\text{Im } \ell_2^+\beta_2}{\text{Re } \beta_2}\}$$

$$+ \frac{1}{2} k^2\alpha_{e2}\{|A^+|^2 \frac{\text{Im } \ell_1^-\beta_1}{\text{Re } \beta_1} + 2 \text{ Im } [A^+ \frac{\beta_1\ell_2^- - \ell_1^+\beta_2^*}{\beta_1+\beta_2^*}] + \frac{\text{Im } \ell_2^-\beta_2}{\text{Re } \beta_2}$$

$$+ \frac{1}{2} k^2 \delta \{|A^+|^2 \frac{\text{Re } \ell_1^+}{\text{Re } \beta_1} + 2 \text{ Re } [A^+ \frac{\ell_1^+ + \ell_2^-}{\beta_1+\beta_2^*}] + \frac{\text{Re } \ell_2^+}{\text{Re } \beta_2}\}$$

and

$$D_5 = |A^+|^2 \frac{1}{2 \text{ Re } \beta_1} + 2 \text{ Re } [A^+ \frac{1}{\beta_1+\beta_2^*}] + \frac{1}{2 \text{ Re } \beta_2}$$

If the exact values for β_1 and β_2 were substituted into Equation 72 the exact value for the propagation velocity could be calculated since R would then take on its stationary value and

$$\frac{R_{\text{stationary}}}{K^2C_{66}} - \alpha = \frac{\rho\omega^2}{K^2C_{66}} - \alpha = \frac{V^2}{V_s^{\,2}} - \alpha$$

where $V = \frac{\omega}{K}$ is the propagation velocity of the surface wave and $V_s = \frac{C_{66}}{\rho}$.

The real power of Equation 72 is in its stationary quality and the fact that it can be used to get a good approximation of velocity with a relatively poor guess of decay constants, β_1 and β_2. Also, as was shown in the last section, the exact value of velocity corresponds to the minimum value of R; and hence, if Equation 72 gives a value for R corresponding to a velocity squared ratio less than

$$\alpha + \frac{\delta^2k^2}{\gamma} - \frac{(\alpha_c\gamma-\alpha_\epsilon x+\alpha_e\delta k^2)^2}{(\gamma-x+4\alpha_c\alpha_\epsilon+2\delta k^2+\delta^2k^2+\alpha_e^{\,2}k^2)(1+k^2)}$$

as dictated by Equation 20, for any value of β_1 and β_2 whatever, then the existence of a surface wave is guaranteed since the true value of velocity will be less than the estimated one.

f. Approximations due to physical considerations A good approximation of Equation 72, which is much simpler in form, can be deduced by way of the following considerations. In all SH-mode piezo-electric surface wave problems solve so far, one of the decay constants has been very small compared to unity; so the real part of β_2 will be assumed very small compared to unity. With this assumption the following become good approximations:

$$\ell_2^+ \approx \frac{\delta}{\gamma}, \quad \mathrm{Im}\,\beta_2 \approx -\frac{\alpha_c\gamma - \alpha_\epsilon x + \alpha_e\delta k^2}{\gamma}$$

and (73)

$$\frac{1}{\mathrm{Re}\,\beta_2} \gg A^+\ell_1^+ \sim A^+\ell_2^+ > A^+$$

With these assumptions the sum of the last term in each of the brackets of Equation 72 can be found. Further, since the last term of the denominator is much larger than the other two, an excellent approximation is obtained by multiplying numerator and denominator by

$$2\,\mathrm{Re}\,\beta_2 - 8\,\mathrm{Re}\,[A^+\frac{1}{\beta_1+\beta_2^*}](\mathrm{Re}\,\beta_2)^2 - 2|A^+|^2\frac{(\mathrm{Re}\,\beta_2)^2}{\mathrm{Re}\,\beta_1}$$

and approximating the denominator by unity. This leads to

$$\frac{V^2}{V_s^2} - \alpha - \frac{\delta^2 k^2}{\gamma} + \left(\frac{\alpha_c \gamma - \alpha_\epsilon \frac{\delta^2 k^2}{\gamma} + \alpha_e \delta k^2}{\gamma} \right)^2$$

$$= (1+k^2)(\text{Re}\,\beta_2)^2 + 2\,\text{Re}\,\beta_2 \{ 2\alpha_c\ \text{Im}\,[A^+ \frac{\beta_1 - \beta_2^*}{\beta_1 + \beta_2^*}]$$

$$+ \alpha_{e1} k^2\ \text{Im}\,[A^+ \frac{\ell_1^+ \beta_1 - \ell_2^- \beta_2^*}{\beta_1 + \beta_2^*}] + \alpha_{e2} k^2\ \text{Im}\,[A^+ \frac{\beta_1 \ell_2^- - \ell_1^+ \beta_2^*}{\beta_1 + \beta_2^*}]$$

$$+ \delta k^2\ \text{Re}\,[A^+ \frac{\ell_1^+ + \ell_2^+}{\beta_1 + \beta_2^*}] + 2[\frac{(\alpha_c \gamma - \alpha_\epsilon \frac{\delta^2 k^2}{\gamma} + \alpha_e \delta k^2}{\gamma^2} - \frac{\delta^2 k^2}{\gamma}]\ \text{Re}\,[A^+ \frac{1}{\beta_1 + \beta_2^*}] \}$$

$$+ |A^+|^2 \{ |\beta_1|^2 \frac{\text{Re}\,\beta_2}{\text{Re}\,\beta_1} + \alpha_c \frac{\text{Im}\,\beta_1}{\text{Re}\,\beta_1} 2\,\text{Re}\,\beta_2 + k^2 |\beta_1|^2 \frac{\text{Re}\,\ell_1^+}{\text{Re}\,\beta_1}\,\text{Re}\,\beta_2$$

$$+ \alpha_{e1} k^2 \frac{\text{Im}\,\ell_1^+ \beta_1}{\text{Re}\,\beta_1}\,\text{Re}\,\beta_2 + \alpha_{e2} k^2 \frac{\text{Im}\,\ell_1^- \beta_1}{\text{Re}\,\beta_1}\,\text{Re}\,\beta_2 + \delta k^2 \frac{\text{Re}\,\ell_1^+}{\text{Re}\,\beta_1}\,\text{Re}\,\beta_2$$

$$+ [\frac{(\alpha_c \gamma - \alpha_\epsilon \frac{\delta^2 k^2}{\gamma} + \alpha_e \delta k^2)^2}{\gamma^2} - \frac{\delta^2 k^2}{\gamma}] \frac{\text{Re}\,\beta_2}{\text{Re}\,\beta_1} \} \qquad (74)$$

It can now be seen that the condition for existence is that the right hand side of Equation 74 be negative. The coefficient of $(\text{Re}\,\beta_2)^2$ in Equation 74 is almost always close to one and is never negative. The value of the third term on the right hand side of Equation 74 is much smaller than the second and will be neglected for now. This leads to the condition of existence that the second term on the right hand side of Equation 74 must be negative, since

$$C = C_1 (\mathrm{Re}\ \beta_2)^2 - 2C_2 (\mathrm{Re}\ \beta_2)$$

where C, C_1, and C_2 are defined by analogy with Equation 74, will always be less than zero for some (Re β_2) if C_1 and C_2 are positive. In fact, the minimum is located at

$$\frac{\partial C}{\partial (\mathrm{Re}\ \beta_2)} = 2C_1\ \mathrm{Re}\ \beta_2 - \beta C_2 = 0 \quad \text{or} \quad \mathrm{Re}\ \beta_2 = \frac{C_2}{C_1} \tag{75}$$

and has a value

$$C = -\frac{C_2^2}{C_1} \tag{76}$$

g. Calculations Equation 74 together with Equation 76 will give the desired velocity from the variational solution and Equation 75 will give the real part of β_2, the smallest decay constant. The imaginary part of β_2 is given by Equation 73. A good approximation for β_1 can be obtained by using the first and third terms of Equation 61. This will give the real part of β_1; the imaginary part will be neglected. The constant C_1 will be approximated by unity, a good approximation since the materials considered have small coupling coefficient. When values for (Re β_2) as given by Equation 75 are tabulated, the values of A^+, $A^+\ell_1^+$, and β_1 will be given for sample cases so that some of the approximations can be checked.

Several examples of the use of Equation 74 will now be given. Figure 3 shows the physical arrangement. It is basically the same as

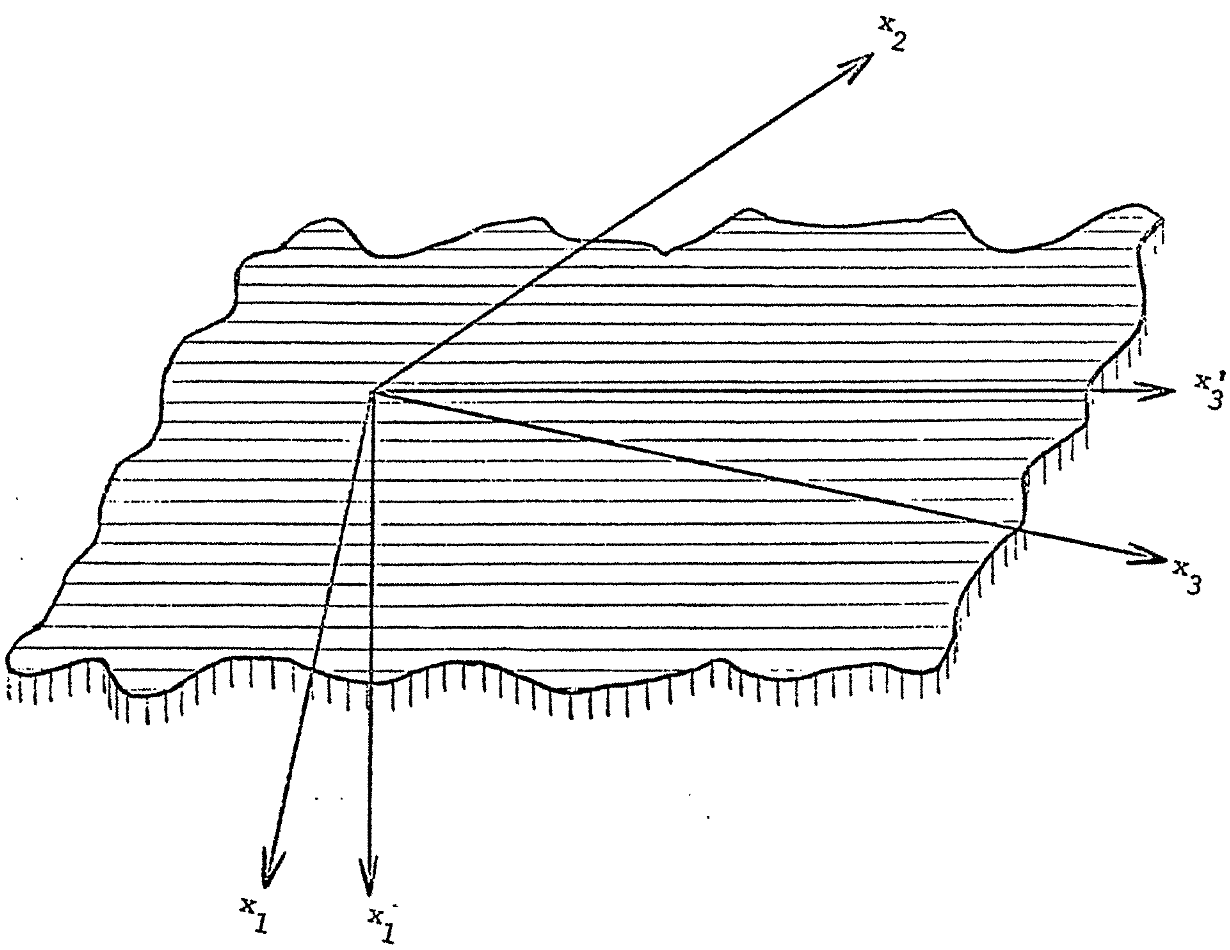

Figure 3. Relation of interface, original coordinate system, and rotated coordinate system

that shown in Figure 2, but the half-space of material is now rotated an angle θ from the original axes. In the figure, the material surface is perpendicular to the x_1' axis. Table 2 gives the parameters for this rotated system for Li_2SO_4. Figures 4 through 11 are graphs of (Re β_2) verses θ where it is positive. The physical constants for the materials were taken from the Landolt-Börnstein (25, 26) tables. The physical constants given in these tables are usually referred to the IEEE standard (19) coordinate system; therefore, the axis of two-fold symmetry will not be parallel to the x_2 axis of Figures 2 and 3 in all cases. When they are not parallel the coordinate system will be transformed to make them so, and the necessary transformation matrix will be given in Figures 4 through 11. In those materials where more than one two-fold axis of symmetry exists, calculations will be given for each different two-fold axis if the results differ from each other.

To obtain the results in Table 2 and in Figures 4 through 11, a program was written for the Hewlett-Packard Model 9100B calculator. That program, which calculates the material constants for the rotated system of Figure 3 and then calculates the parameters given in the table and figures from Equation 74, is given in detail with operating instructions in Appendix A. When (Re β_2), as calculated from Equation 74, is positive the surface wave can exist and the value given in the figures is the real part of the smallest decay constant. When (Re β_2) in the table is negative the surface wave cannot exist.

Table 2. Various parameters from the variational calculation of Re β_2 for $Li_2SO_4 \cdot H_2O$

θ	Re β_2	Re A^+	Im A^+	Re $A^+ \ell_1^+$	Im $A^+ \ell_1^+$	β_1
70	.0024	.0132	.0067	.358	.2883	.918
90	.0003	-.0092	.0014	- .955	1.955	.924
110	.0186	-.0877	-.0052	-1.053	3.175	.915
130	.0838	-.1215	-.0854	- .353	1.821	.948
160	.0143	.0330	-.0210	- .180	.488	1.068

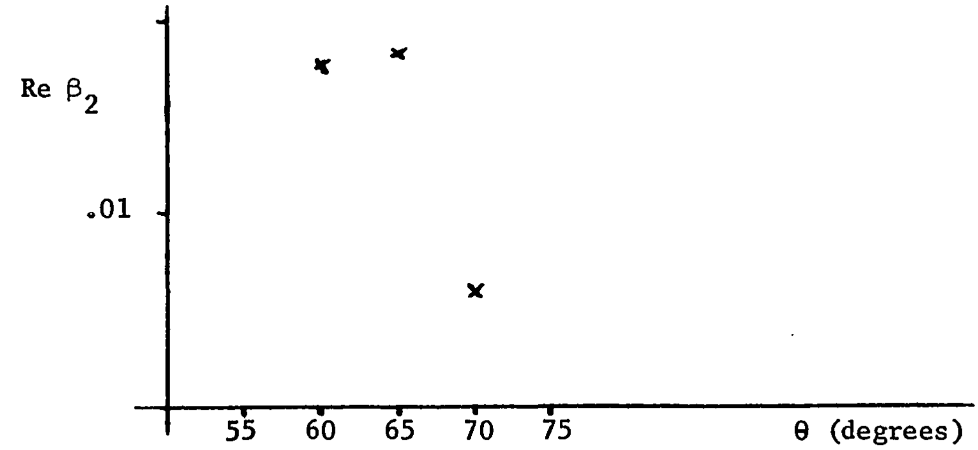

Figure 4. Rochelle salt. 222. $a_{ij} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(The Re β_2 is symmetrical about $\theta = 90^{\circ}$)

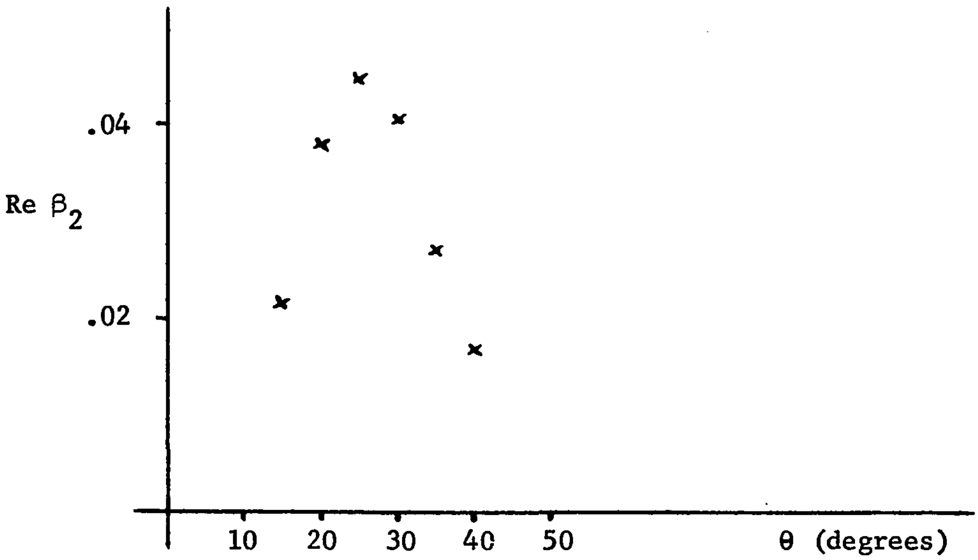

Figure 5. Rochelle salt. 222. $a_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(The Re β_2 is symmetrical about θ 90°)

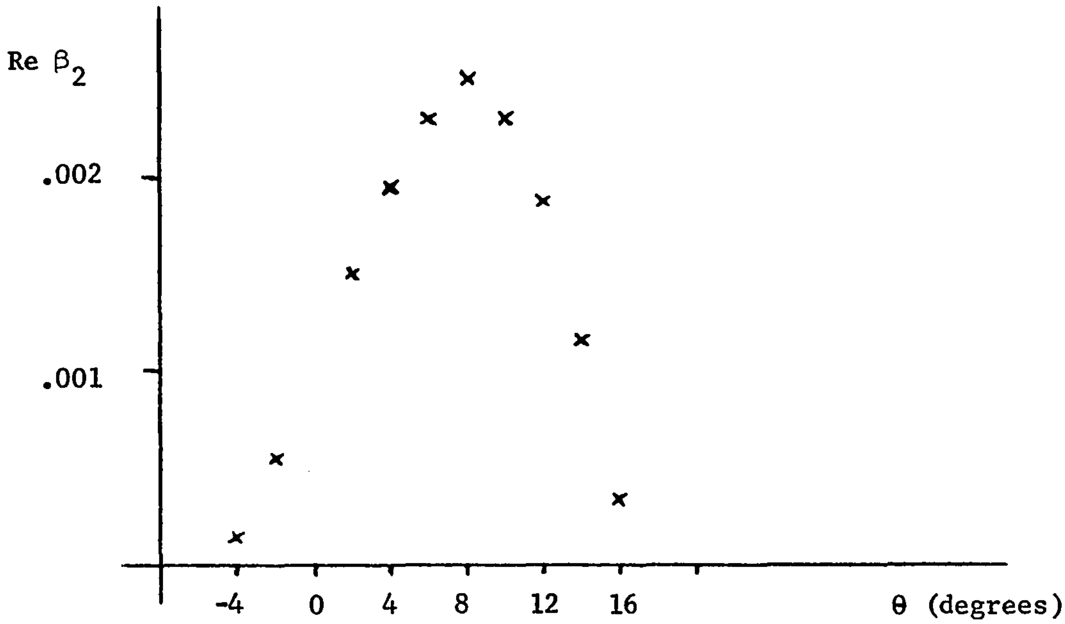

Figure 6. Quartz, 32. $a_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

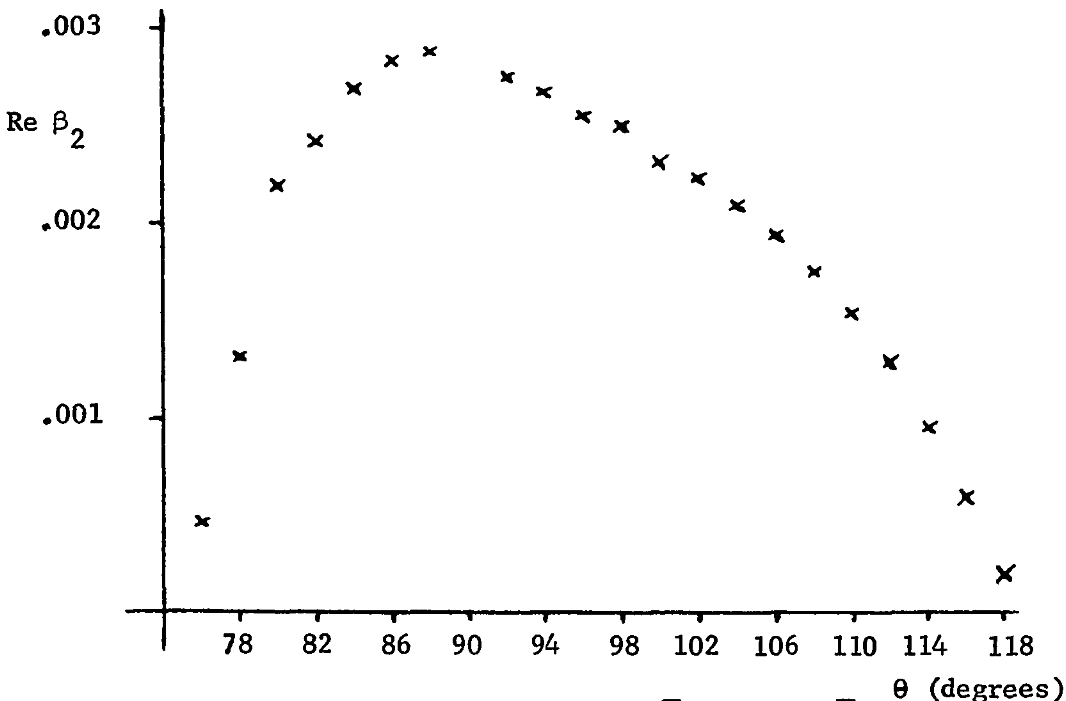

Figure 7. Quartz, 32. $a_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

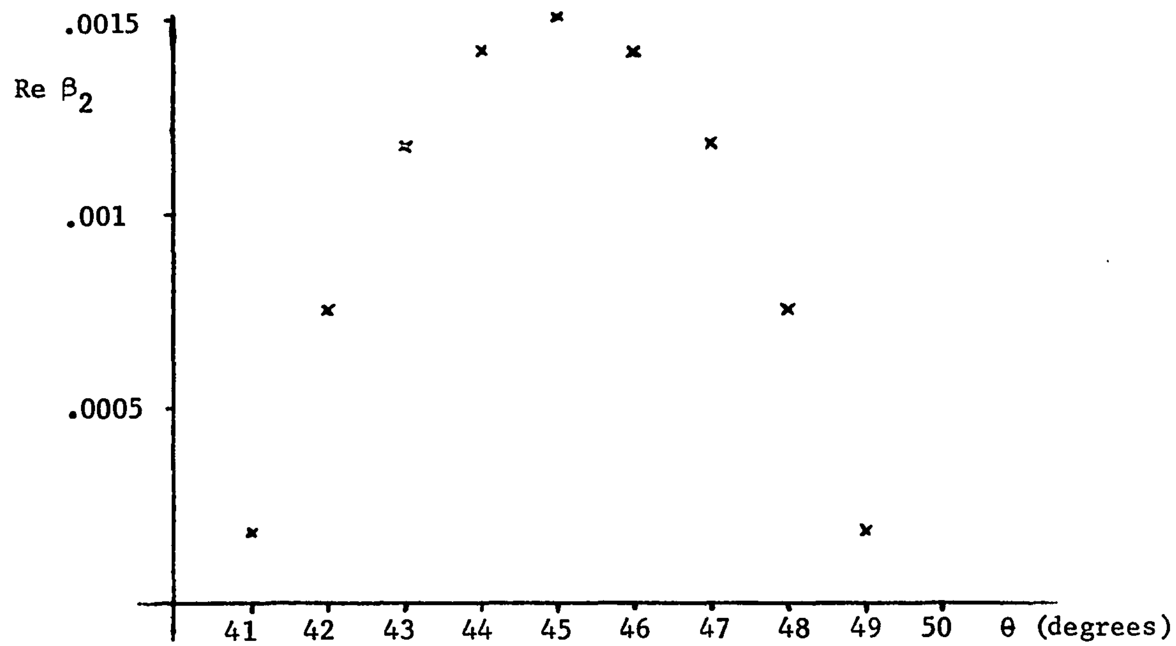

Figure 8. $Bi_{12}GeO_{20}$. 23

(See Table 1 for equivalent directions)

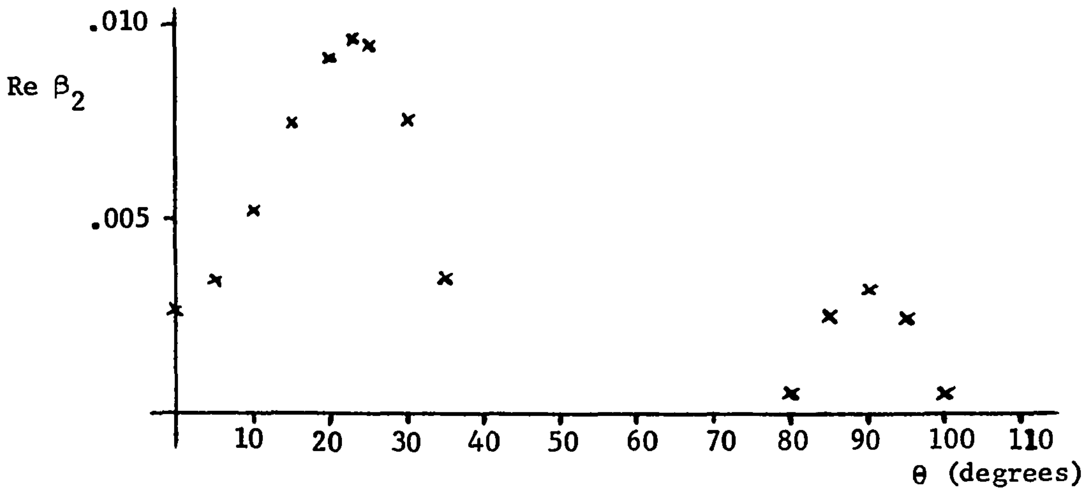

Figure 9. Lithium gallium oxide. mm2. $a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

(The Re β_2 is symmetrical about $\theta = 90^o$)

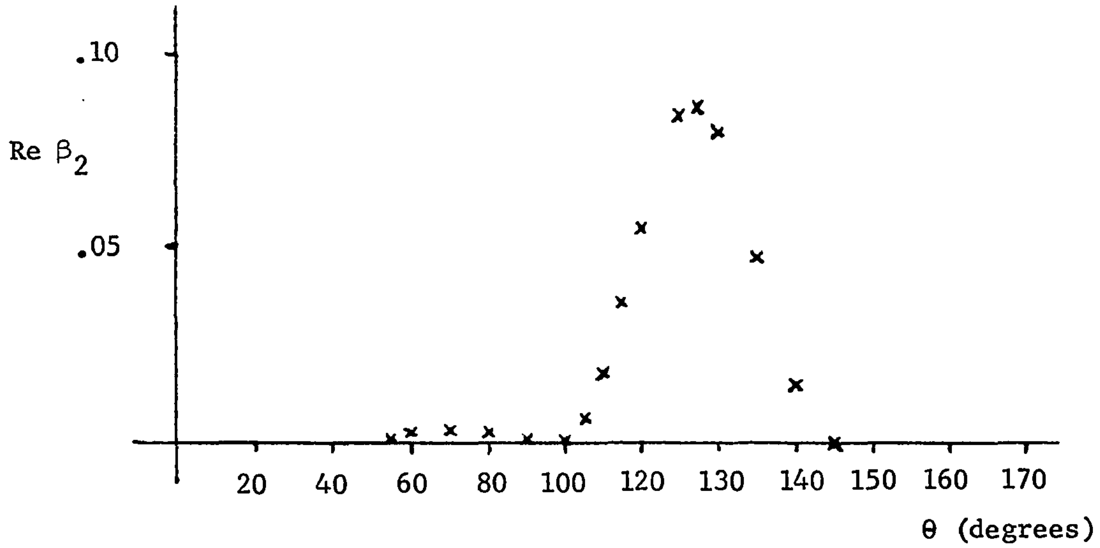

Figure 10. $Li_2SO_4 \cdot H_2O$. 2

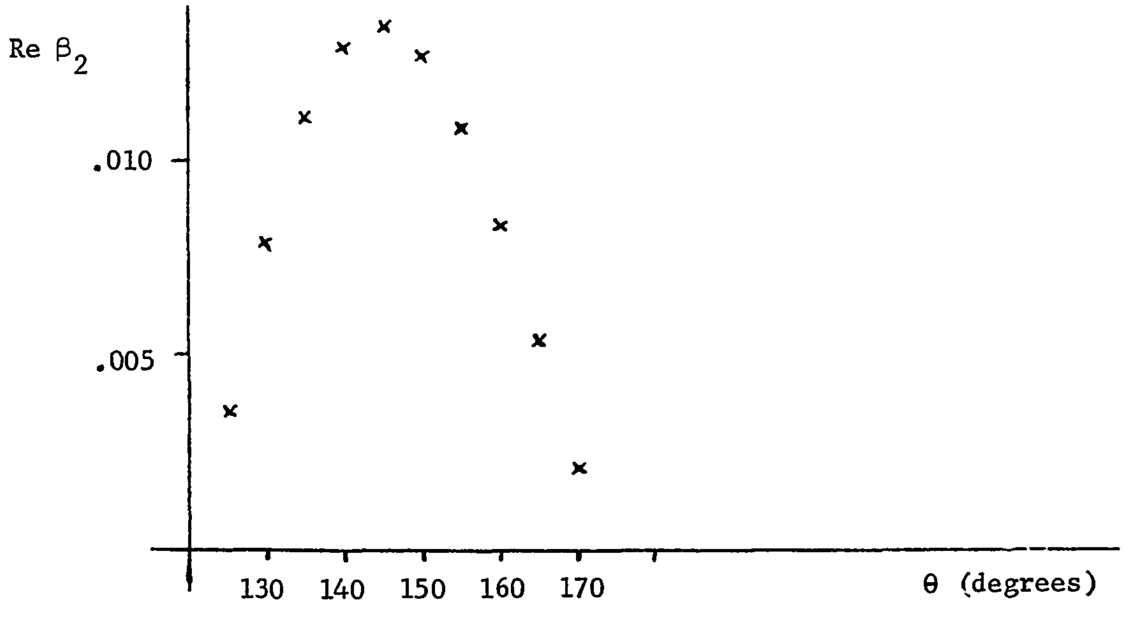

Figure 11. EDT. 2

V. CONCLUSIONS

The methods of this thesis should allow the ultrasonics engineer to optimize the parameters of his devices within the limits of available materials. As can be seen from the sample calculations, some of the materials in 222, 23, and 2 symmetry apparently have much smaller penetrations at certain angles than any of the cases solved previously. This should make a considerable difference in the efficiency of transduction with interdigital electrodes, and in the available gain of a seperate medium amplifier.

The penetration depth is a parameter which may have to be selected for another device which hasn't been developed yet. In 1967, A. E. Lord (28) considered the possibility of parametric amplification of a transverse wave with a longitudinal wave or vice-versa. In his article he treated only bulk waves and concluded that the amplitudes needed for this interaction were impractical. With surface waves the amplitudes are inherently larger and in fact the amplitude levels used by Slobodnik (36) in his experiments with harmonic generation were considerably larger than what Lord suggested was needed for efficient parametric interaction. If a surface wave device of this type is made it will require a matching of penetration depths of the two waves for maximum efficiency.

If extremely accurate calculations of velocity or fields is needed, they could be found rapidly by substituting the value of velocity from the variational solution into an iterative program. The iteration in such a program has to be carried out on a multivariable basis (27) since there are four variables to be found: Re β_1, Im β_1, Re β_2, and Im β_2.

The variational development presented should be applicable to piezoelectric wave guides. The parameters in the formulae would be various form factors for the particular guide instead of decay constants, and the criterion for existence of the waves would be some other limit on the phase velocity; but the rest of the formulism should be very similar.

All the examples given were for a free surface. The case of a lightly metalized surface is a simplification of this case and values for it can be found by letting the permittivity of the non-material side of the interface go to infinity. When this is done it is seen that the range of existence is generally increased and in some cases will exist for an arbitrary rotation about the two-fold axis.

VI. LITERATURE CITED

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VIII. APPENDIX

The program for the Hewlett-Packard Model 9100B calculator is in three parts and stored on three different magnetic storage cards. These three parts are entered and run consecutively and after the last one is run the negative of (Re β_2) appears in the y register.

The detailed instructions are as follows:

Settings: Angle - Degrees, Mode - Run.

Press GO TO ( )( )

-

0

0

With card with Program A on it in card reader

Press Enter

Press GO TO ( )( )

-

0

0

Enter the following parameters in order from the keyboard and after each one press CONT: θ, e_{14}, $-e_{16}$, e_{34}, $-e_{36}$, ϵ_{13}, $-\epsilon_{11}$, ϵ_{33}, $-\epsilon_{13}$, C_{46}, $-C_{66}$, C_{44}, $-C_{46}$. Enter Program B the same way as Program A and run it by pressing CONT. Enter Program C and run it by pressing CONT.

When Program A has run, the material parameters are located as shown in the following table.

Parameter	Location
α_{e1}	6
α_{e2}	5
δ	4
α_{ϵ}	2
γ	1
α_{c}	a
α	9
k^2	8
ϵ_{11r}	0

When Program B has run, the material parameters and defined quantities are located as follows.

Quantity	Location
α_{e1}	6
δ	4
α_{ϵ}	2
γ	1
k^2	8
α_{c}	a
Re A^+	b
Im A^+	7
Re $A^+ \ell_1^+$	x
Im $A^+ \ell_1^+$	y
β_1	C
α_{e}	0

Program B calculates β_1 from the formula

$$\beta_1^2 = \gamma - \delta^2 k^2 + 2\delta k^2 + \alpha_e^2 k^2 + 4\alpha_\varepsilon \alpha_c - \gamma k^2$$

A. Program A (Rotates Axis)

```
x → ( )
  f
GO TO ( )( )
SUB/RET
  -
  6
  2
  a
x ⇄ y
  ÷
y → ( )
  6
  ↑
x ← ( )
  9
x ⇄ y
  ÷
y → ( )
  5
  ↑
x ← ( )
  8
x ⇄ y
  ÷
y → ( )
  4
x ← ( )
  7
  ↑
  x
y → ( )
  3
GO TO ( )( )
SUB/RET
  —
  6
  2
  a
x ⇄ y
  ÷
y → ( )
  2
  ↑
x ← ( )
  8
x ⇄ y
  ÷
y → ( )
  1
  ↑
x ← ( )
  3
x ⇄ y
  ÷
y → ( )
  3
x → ( )
  0
GO TO ( )( )
SUB/RET
```

```
—
6
2
a
x ⇄ y
÷
y → ( )
a
↑
x ← ( )
8
x ⇄ y
÷
y → ( )
9
↑
x ← ( )
3
x ⇄ y
÷
1
1
3
x
y → ( )
8
```

1. First subroutine

```
STOP
x → ( )
e
STOP
x → ( )
d
STOP
x → ( )
c
STOP
x → ( )
b
GO TO ( )( )
SUB/RET
—
b
a
y → ( )
a
e
chg sign
↑
y ⇄ ( )
b
↓
chg sign
x → ( )
e
GO TO ( )( )
SUB/RET
—
b
a
y → ( )
9
e
chg sign
↑
y ⇄ ( )
d
↓
chg sign
↑
y ⇄ ( )
b
y ⇄ ( )
c
y ⇄ ( )
```

e

GO TO ( )( )

SUB/RET

−

b

a

y → ( )

8

e

↑

y ⇄ ( )

b

y → ( )

e

d

chg sign

x → ( )

d

c

chg sign

x → ( )

c

GO TO ( )( )

SUB/RET

−

b

a

y → ( )

7

SUB/RET

<u>2. Second subroutine</u>

f

cos x

↑

x

e

x

f

sin x

↑

f

cos x

x

d

x ⇄ y

x

Roll ↓

+

c

Roll ↑

x

Roll ↓

+

f

sin x

↑

x

b

x

Roll ↓

+

SUB/RET

B. Program B (Calculates A^{+} and $A^{+}\ell_1^{+}$)

2
↑
x ← ( )
4
x
↑
x
Roll ↓
−
x ← ( )
6
↑
x ← ( )
5
+
y → ( )
7
↓
↑
x
Roll ↓
+
x ← ( )

1
−
x ← ( )
8
x
x ← ( )
2
↑
a
x
4
x
Roll ↓
+
x ← ( )
1
+
y → ( )
b
↑
↓
x ← ( )
4

−
Roll ↑
√x
x → ( )
c
x ⇄ y
x ← ( )
7
x
Roll ↓
x ⇄ y
TO POLAR
ln x
y → ( )
e
x → ( )
f
b
↑
x ← ( )
1
−
c

↑

2

x

x ← ( )

2

x

↓

x ⇄ y

TO POLAR

ln x

ACC -

RCL

x → ( )

d

y → ( )

5

c

↑

x ← ( )

0

x → ( )

b

x ← ( )

7

x → ( )

0

b

x

1

+

x ← ( )

2

↑

b

x

Roll ↓

x ⇄ y

TO POLAR

ln x

ACC +

x ← ( )

6

↑

b

x

y → ( )

7

c

↑

b

x

y → ( )

3

RCL

e^x

TO RECT

y → ( )

e

x → ( )

f

↓

x ← ( )

3

ACC -

RCL

TO POLAR

ln x

y → ( )

3

x → ( )

9

x ← ( )

2

↑

b

x
x ← ( )
4
↑
x ← ( )
1
÷
↓
x
chg sign
Roll ↑
x ← ( )
7
Roll ↑
−
Roll ↓
x ⇄ y
TO POLAR
ln x
x → ( )
f
y → ( )
e

x ← ( )
3
↑
x ← ( )
9
ACC −
RCL
e^x
TO RECT
x → ( )
b
y → ( )
7
x ← ( )
5
↑
d
ACC +
RCL
e^x
TO RECT
STOP

C. Program C (Calculates Re β_2)

y → ( )	↑	a
d	x	↑
↑	x ← ( )	x ← ( )
x ← ( )	1	1
4	÷	x
x	b	x ← ( )
x ← ( )	x	2
8	x ← ( )	↑
x	8	x ← ( )
c	x	4
÷	↓	x
Roll ↑	+	x ← ( )
x ⇄ y	2	4
x ← ( )	↑	x
6	x ← ( )	x ← ( )
x	7	8
x ← ( )	x	x
8	a	x ← ( )
x	x	1
↓	↓	÷
+	+	↓
x ← ( )	y → ( )	−
4	3	x ← ( )

0
↑
x ← ( )
4
x
x ← ( )
8
x
↓
+
x ← ( )
1
÷
x ⇄ y
↑
x
x ← ( )
4
↑
x
x ← ( )
8

x
x ← ( )
1
÷
↓
−
2
↑
b
x
c
÷
↓
x
x ← ( )
3
+
STOP